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Admissible invariants of genus 3 curves

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Abstract. Several invariants of polarized metrized graphs and their applications in Arithmetic Geometry are studied recently. In this paper, we explicitly calculated these admissible invariants for all curves of genus 3. We find the sharp lower bound for the invariants φ, λ and ϵ for all polarized metrized graphs of genus 3. This improves the lower bound given for Effective Bogomolov Conjecture for such curves.

1. Introduction

Invariants of a polarized metrized graph $(\Gamma, \mathbf{q})$ are of interest for more than last twenty years because of their applications in arithmetic geometry and number theory. The invariants $\varphi(\Gamma)$, $\lambda(\Gamma)$ and $\epsilon(\Gamma)$, see below for their definitions, are studied for their connection to the self intersection of admissible dualizing sheaf associated to a curve of genus at least 2 over a global field. One can consult to S. Zhang's articles [24] and [25] for technical details. On the other hand, L. Szpiro showed in [19] and [20] that the Bogomolov conjecture and the effective Mordell conjecture can be stated as 'suitable' lower and upper bounds to self intersection of certain dualizing sheaf, respectively.

Lower bounds for $\varphi(\Gamma)$, $\lambda(\Gamma)$ are given in [7, Theorems 2.11 and 2.13] for all curves of genus greater than 1. However, obtaining sharp bounds and explicit computations for each possible case were done for only certain type of curves. For curves of small genus, the invariants $\varphi(\Gamma)$, $\lambda(\Gamma)$ and $\epsilon(\Gamma)$ are studied in [11,12, 16,17,21,23]. It will be of reader's interest to see [13,14] and [22], too.

When $(\Gamma, \mathbf{q})$ has genus 2, Moriwaki computed several invariants including $\epsilon(\Gamma)$ for each case and obtained certain bound equivalent to the sharp lower bound $\varphi(\Gamma) \geq \frac{1}{27}\ell(\Gamma)$, where $\ell(\Gamma)$ is the total length of Γ. This bound is verified with different methods by X. Faber in [11] and the author in [7]. For such a Γ, Moriwaki in [17] and R. de Jong in [12] explicitly computed these invariants for each genus

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2 curves. De Jong's work also extends to archimedean case, which is essential for number theoretic applications.

When $(\Gamma, \mathbf{q})$ has genus 3, Yamaki in [23], Faber in [11] and the author in [7] studied these invariants. Moreover, Faber showed [11, Theorem 3.4] that $\varphi(\Gamma) \geq c\delta_0(\Gamma) + \frac{4}{3}\delta_1(\Gamma)$ with $c = \frac{2}{81}$. Later, the author in [7, Theorem 2.11] improved this result by showing that c can be taken as $\frac{1}{39}$. However, these lower bounds are not sharp.

Contributions of this paper are as follows:

(i) We determine all polarized metrized graphs $(\Gamma, \mathbf{q})$ of genus 3.

(ii) We explicitly compute various invariants including $\varphi(\Gamma)$, $\epsilon(\Gamma)$ and $\lambda(\Gamma)$ of all $(\Gamma, \mathbf{q})$ of genus 3.
We think that the results in parts (i) and (ii) can be determined by the experts working in this area, but these explicit results were not available in the literature previously.

(iii) An arithmetic version of Bogomolov-Miyaoka-Yau inequality is formulated by Parshin [18] and Moret-Bailly [15] for arithmetic surfaces. This inequality is known to be equivalent to Effective Mordell conjecture, Szpiros's discriminant conjecture and ABC conjecture. Zhang's work [25] shows that this inequality can be stated as an inequality involving the self intersection of Gross-Schoen cycle and certain invariants φ_v and ϵ_v depending on both archimedean and non-archimedean places v's. For a non-archimedean place, these invariants are nothing but the invariants $\varphi(\Gamma)$ and $\epsilon(\Gamma)$. One possible approach to prove these important conjectures is to prove the inequality implied by Zhang's work. It will be interesting and informative to verify this inequality for curves of small genus. In this framework, our work in this paper for genus 3 case and de Jong's work [12] for genus 2 case fill certain important gaps.

(iv) We give sharp lower bounds for each of $\varphi(\Gamma)$, $\epsilon(\Gamma)$ and $\lambda(\Gamma)$. Namely, we show that if $(\Gamma, \mathbf{q})$ is of genus 3 and of total length $\ell(\Gamma)$, then $\varphi(\Gamma) \geq \frac{17}{288}\ell(\Gamma)$, $\lambda(\Gamma) \geq \frac{3}{28}\ell(\Gamma)$ and $\epsilon(\Gamma) \geq \frac{2}{9}\ell(\Gamma)$. As a result, via Zhang's work, this improves the bound given for Effective Bogomolov Conjecture for genus 3 case (see [7, Theorems 2.3 and 2.4]). Note that the bound $\frac{17}{288}\ell(\Gamma)$ for $\varphi(\Gamma)$ was conjectured by Faber in [11, Remark 5.1]. This lower bound is attained when Γ is a regular tetrahedral graph.

(v) We obtain, as a byproduct, a highly nontrivial inequality that holds for any nonnegative six real numbers (see inequalities (6), (7) and (9) below). Interestingly, the terms that appear in these inequalities correspond to certain cycles of a tetrahedral graph which is not necessarily in $\mathbb{R}^3$.

2. Pm-graphs and their invariants

In this section, we first give brief descriptions of a metrized graph Γ, a polarized metrized graph $(\Gamma, \mathbf{q})$, invariants $\tau(\Gamma)$, $\theta(\Gamma)$, $\epsilon(\Gamma)$, $\varphi(\Gamma)$, $\lambda(\Gamma)$ and $Z(\Gamma)$ associated to $(\Gamma, \mathbf{q})$.

A metrized graph Γ is a finite connected graph equipped with a distinguished parametrization of each of its edges. A metrized graph Γ can have multiple edges and self-loops. For any given $p \in \Gamma$, the number $\upsilon(p)$ of directions emanating from p will be called the *valence* of p. By definition, there can be only finitely many $p \in \Gamma$ with $\upsilon(p) \neq 2$.

For a metrized graph Γ, we will denote a vertex set for Γ by $V(\Gamma)$. We require that $V(\Gamma)$ be finite and non-empty and that $p \in V(\Gamma)$ for each $p \in \Gamma$ if $\upsilon(p) \neq 2$. For a given metrized graph Γ, it is possible to enlarge the vertex set $V(\Gamma)$ by considering additional valence 2 points as vertices.

For a given metrized graph Γ with vertex set $V(\Gamma)$, the set of edges of Γ is the set of closed line segments or loops with end points in $V(\Gamma)$. We will denote the set of edges of Γ by $E(\Gamma)$. However, if e_i is an edge, by $\Gamma - e_i$ we mean the graph obtained by deleting the interior of e_i.

We define the genus of Γ to be the first Betti number $g(\Gamma) := e - \upsilon + 1$ of the graph Γ, where e and υ are the number of edges and vertices of Γ, respectively.

The length of an edge of Γ is a positive real number. If we denote the length of an edge $e_i \in E(\Gamma)$ by L_i, the total length of Γ, which is denoted by $\ell(\Gamma)$, is given by $\ell(\Gamma) = \sum_{i=1}^{e} L_i$.

The tau constant $\tau(\Gamma)$ of a metrized graph Γ was initially defined by Baker and Rumely in [2, Section 14]. The following lemma gives a description of the tau constant. In particular, it implies that the tau constant is positive.

Lemma 2.1. *[2, Lemma 14.4] Let $r(x, y)$ be the resistance function on Γ. For any fixed y in Γ, $\tau(\Gamma) = \frac{1}{4}\int_\Gamma \left(\frac{\partial}{\partial x} r(x, y)\right)^2 dx$.*

One can find more detailed information on $\tau(\Gamma)$ in articles [4–6] and [8]. For more information about the resistance function $r(x, y)$ on a metrized graph, one can consult to the articles [2], [1] and [5].

Let Γ be a metrized graph with a vertex set $V(\Gamma)$, and let $\mathbf{q} : \Gamma \to \mathbb{N}$ be a function supported on a subset of $V(\Gamma)$. That is, $\mathbf{q}(s) = 0$ for all $s \in \Gamma - V(\Gamma)$, and whenever $\mathbf{q}(s) > 0$ we must have $s \in V(\Gamma)$.

A divisor on Γ is a formal sum $\sum n_i p_i$, where $a_i \in \mathbb{Z}$ and $p_i \in \Gamma$ for every i. A divisor $\sum n_i p_i$ on Γ is called effective if $n_i \geq 0$ for all i.

The canonical divisor K of $(\Gamma, \mathbf{q})$ is defined as follows:

$$K = \sum_{p \in V(\Gamma)} (\upsilon(p) - 2 + 2\mathbf{q}(p))p. \tag{1}$$

The pair $(\Gamma, \mathbf{q})$ is called a *polarized metrized graph* (pm-graph in short) if K is an effective divisor. Whenever $\mathbf{q} = 0$, $(\Gamma, \mathbf{q})$ is called a simple pm-graph. We define the *genus* $\bar{g}(\Gamma)$ of a pm-graph $(\Gamma, \mathbf{q})$ as follows:

$$\bar{g}(\Gamma) = g(\Gamma) + \sum_{p \in V(\Gamma)} \mathbf{q}(p). \tag{2}$$

If Γ under consideration is clear, we simply use notations g and $\bar{g}$ instead of $\bar{g}(\Gamma)$ and $g(\Gamma)$, respectively.

Remark 2.2. For each $p \in V(\Gamma)$, $\upsilon(p) - 2 + 2\mathbf{q}(p) \geq 0$ and $\mathbf{q}(p) \geq 0$, since the canonical divisor K is effective and $\mathbf{q}$ is nonnegative. In particular, if $\upsilon(p) = 1$ for some $p \in \Gamma$, we should have $p \in V(\Gamma)$ and $\mathbf{q}(p) \geq 1$.

On a pm-graph $(\Gamma, \mathbf{q})$, we defined and studied the invariant $\theta(\Gamma)$ in [4] and [7] as follows:

$$\theta(\Gamma) = \sum_{p,\, q \in V(\Gamma)} (\upsilon(p) - 2 + 2\mathbf{q}(p))(\upsilon(q) - 2 + 2\mathbf{q}(q)) r(p, q). \tag{3}$$

We have $\theta(\Gamma) \geq 0$ for any pm-graph Γ, since the canonical divisor K is effective.

Let $\mu_{ad}(x)$ be the admissible measure associated to K (defined by Zhang [24, Lemma 3.7]). Next, we give definitions of the invariants $\epsilon(\Gamma)$, $\varphi(\Gamma)$, $\lambda(\Gamma)$ and $Z(\Gamma)$ (c.f. [25, Section 4.1]) of Γ:

$$\begin{aligned}
\epsilon(\Gamma) &= \iint_{\Gamma \times \Gamma} r(x, y) \delta_K(x) \mu_{ad}(y), \\
Z(\Gamma) &= \frac{1}{2} \iint_{\Gamma \times \Gamma} r(x, y) \mu_{ad}(x) \mu_{ad}(y), \\
\varphi(\Gamma) &= 3\bar{g} \cdot Z(\Gamma) - \frac{1}{4}(\epsilon(\Gamma) + \ell(\Gamma)), \\
\lambda(\Gamma) &= \frac{\bar{g} - 1}{6(2\bar{g} + 1)} \varphi(\Gamma) + \frac{1}{12}(\epsilon(\Gamma) + \ell(\Gamma)).
\end{aligned} \tag{4}$$

We can express each invariant given in Equation (4) in terms of $\tau(\Gamma)$ and $\theta(\Gamma)$ ([7, Propositions 4.6, 4.7, 4.9 and Theorem 4.8]):

Theorem 2.3. *Let $(\Gamma, \boldsymbol{q})$ be a pm-graph with $\bar{g} = 3$. Then we have*

$$\varphi(\Gamma) = \frac{13}{3}\tau(\Gamma) + \frac{\theta(\Gamma)}{12} - \frac{\ell(\Gamma)}{4}, \qquad Z(\Gamma) = \frac{5}{9}\tau(\Gamma) + \frac{\theta(\Gamma)}{72},$$
$$\lambda(\Gamma) = \frac{3}{7}\tau(\Gamma) + \frac{\theta(\Gamma)}{56} + \frac{\ell(\Gamma)}{14}, \qquad \epsilon(\Gamma) = \frac{8}{3}\tau(\Gamma) + \frac{\theta(\Gamma)}{6}.$$

Let p be a point in a pm-graph Γ such that $p \notin V(\Gamma)$. That is, p is not an end point of any edge in $E(\Gamma)$. Let Γ-p be the pm-graph obtained from Γ by removing p and adding two vertex points p_1 and p_2 to make the remaining parts closed as illustrated in Fig. 1. Following Zhang's definition [25, Section 4.1], we call p is of type 0 if Γ-p is connected. This happens when p is contained in an edge such that removing the edge does not disconnect Γ. If p is not of type 0, Γ-p is a union of two connected metrized subgraphs with functions $\mathbf{q}_1$ and $\mathbf{q}_2$ that are restrictions of $\mathbf{q}$ and satisfy $\mathbf{q}_1(p_1) = \mathbf{q}_2(p_2) = 0$. By applying Equation (2), we see that the subgraphs are of genus i and $\bar{g} - i$ for some integer $i \in (0, \bar{g}/2]$. In this case, we call p is of type i. For each integer $i \in [0, \bar{g}/2]$, let Γ_i be the subgraph of Γ of points of type i, and let $\ell_i(\Gamma)$ be the total length of Γ_i. We use the invariants $\delta_i(\Gamma) := \ell_i(\Gamma)$ for each $i \geq 0$ (see [7] or [25] for geometric meaning of these invariants). Therefore, whenever $\bar{g} = 3$, we have only type 0 and

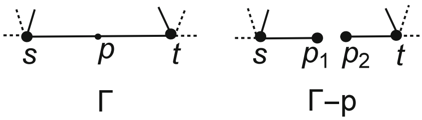

Fig. 1. Γ and Γ-p, where $p \notin V(\Gamma)$

1 points, so we consider only the invariants $\delta_0(\Gamma)$ and $\delta_1(\Gamma)$ for which we have $\ell(\Gamma) = \delta_0(\Gamma) + \delta_1(\Gamma)$.

Remark 2.4. Given a pm-graph $(\Gamma, \mathbf{q})$ with a vertex set $V(\Gamma)$ containing at least two elements, suppose $\mathbf{q}(s) = 0$ and $\upsilon(s) = 2$ for some $s \in V(\Gamma)$. Then removing s from the vertex set of Γ does not change $\theta(\Gamma)$. Similarly, if a vertex s is such that $\upsilon(s) = 2$ and $V(\Gamma) - \{s\}$ has at least one element, then removing s from $V(\Gamma)$ does not change $\tau(\Gamma)$. (such vertices are called eliminable vertices in [21, pg. 152]). We call these the *valence property* of $\tau(\Gamma)$ and $\theta(\Gamma)$, see [5, Remark 2.10]). Therefore, $\epsilon(\Gamma)$, $Z(\Gamma)$, $\varphi(\Gamma)$ and $\lambda(\Gamma)$ do not change under this process by Theorem 2.3. That is, each of these invariants has the valence property [9, Remark 2.4].

Remark 2.4 is very helpful to determine the possible pm-graphs of a given genus. As long as $V(\Gamma)$ is non-empty, it will be enough to consider pm-graphs not having any vertex p with $\upsilon(p) = 2$ and $\mathbf{q}(p) = 0$. In fact, in this way we choose only one model of a given pm-graph among all the equivalent models.

Recall that we consider only pm-graphs with $\bar{g} = 3$ in this paper. Such pm-graphs can have any $g \in \{0, 1, 2, 3\}$ by Equation (2). We designate a section for each such value of g in the rest of the article. In each case, we first determine the pm-graphs with the desired g and $\bar{g}$ up to equivalence. Then, we compute all the relevant invariants. Finally, we find sharp lower bounds to the invariants $\tau(\Gamma)$ and $\varphi(\Gamma)$, $\lambda(\Gamma)$ for all Γ under consideration. We can use the techniques developed in [5] and [4] along with Theorem 2.3 to compute these invariants. This is what we did in this paper. Alternatively, we can compute these invariants by using the algorithms given in [8] and [9]. For example, how we compute the tau constant for the metrized graph in part IX of Fig. 5 is illustrated in [8, Example 5.2], and computation of invariants of the pm-graph in part XIV of Fig. 5 are done in [9, Example 1].

Note that we could consider only pm-graphs $XIII$ and XIV in Fig. 5 to obtain lower bound of $\varphi(\Gamma)$ for all pm-graphs of $\bar{g} = 3$. The proof of this fact was given in [11, pages 360 and 365] (see also [7, pages 549 and 550]). Another approach utilizing this fact was also known to Yamaki (see [23, page 67] and [21, page 160]). However, our aim in this paper is more than finding the sharp lower bound, so we worked on all possible pm-graphs.

3. The case $g(\Gamma) = 0$

Suppose Γ is a pm-graph with $g(\Gamma) = 0$ and $\bar{g}(\Gamma) = 3$. Then Equation (2) becomes $3 = \sum_{p \in V(\Gamma)} \mathbf{q}(p)$. Moreover, it follows from Remark 2.2 that such a pm-graph

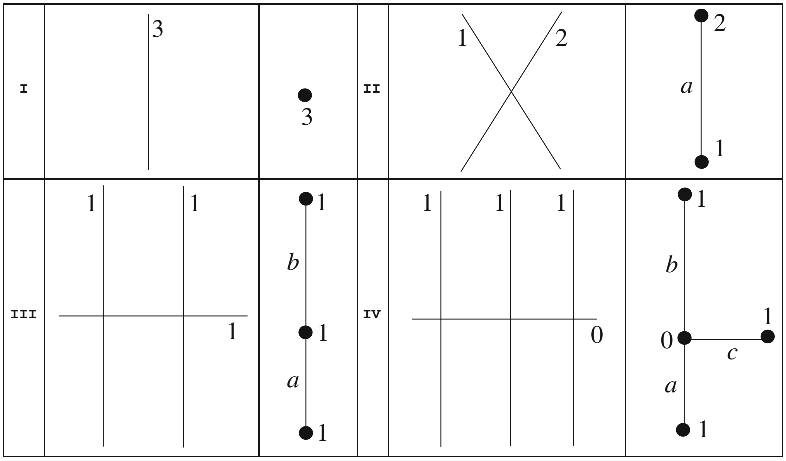

Fig. 2. Irreducible components of genus 3 curves and their dual graphs when the dual graphs have genus 0, i.e. when $\bar{g} = 3$ and $g = 0$

Table 1. Pm-graph invariants when $g(\Gamma) = 0$

	$\ell(\Gamma)$	$\delta_0(\Gamma)$	$\delta_1(\Gamma)$	$\tau(\Gamma)$	$\theta(\Gamma)$	$\varphi(\Gamma)$	$\lambda(\Gamma)$	$\epsilon(\Gamma)$
I	0	0	0	0	0	0	0	0
II	a	0	$\ell(\Gamma)$	$\frac{\ell(\Gamma)}{4}$	$6\ell(\Gamma)$	$\frac{4\ell(\Gamma)}{3}$	$\frac{2\ell(\Gamma)}{7}$	$\frac{5\ell(\Gamma)}{3}$
III	$a+b$	0	$\ell(\Gamma)$	$\frac{\ell(\Gamma)}{4}$	$6\ell(\Gamma)$	$\frac{4\ell(\Gamma)}{3}$	$\frac{2\ell(\Gamma)}{7}$	$\frac{5\ell(\Gamma)}{3}$
IV	$a+b+c$	0	$\ell(\Gamma)$	$\frac{\ell(\Gamma)}{4}$	$6\ell(\Gamma)$	$\frac{4\ell(\Gamma)}{3}$	$\frac{2\ell(\Gamma)}{7}$	$\frac{5\ell(\Gamma)}{3}$

can have at most three vertices with valence exactly 1. On the other hand, $g(\Gamma) = 0$ implies $e = \upsilon - 1$.

If Γ has no edges, then it is just a point p with $\mathbf{q}(p) = 3$, as shown in part *I* of Fig. 2. Otherwise, pm-graphs $(\Gamma, \mathbf{q})$ in this case are tree pm-graphs, which have at least two vertices with $\upsilon(p) = 1$. Figure 2 illustrates the possible pm-graphs satisfying these conditions.

We have $\tau(\Gamma) = \frac{\ell(\Gamma)}{4}$ since Γ is a metrized graph that is a tree graph [2, Eq. 14.3], and we use Equation (3) to compute $\theta(\Gamma)$. Then we compute $\varphi(\Gamma)$, $\lambda(\Gamma)$ and $\epsilon(\Gamma)$ by using Theorem 2.3. The results are given in Table 1.

We exclude the case *I* in Table 1 as $\ell(\Gamma) = 0$. In the other three cases, we have $\varphi(\Gamma) = \frac{4}{3}\ell(\Gamma)$, $\lambda(\Gamma) = \frac{2\ell(\Gamma)}{7}$ and $\epsilon(\Gamma) = \frac{5\ell(\Gamma)}{3}$.

4. The case $g(\Gamma) = 1$

In this section, we consider pm-graphs $(\Gamma, \mathbf{q})$ with $g = 1$ and $\bar{g} = 3$. It follows from Equation (2) that $2 = \sum_{p \in V(\Gamma)} \mathbf{q}(p)$. Thus, such a pm-graph can have at most two vertices with valence exactly 1 by Remark 2.2. Since $g = 1$, we have $e = \upsilon$.

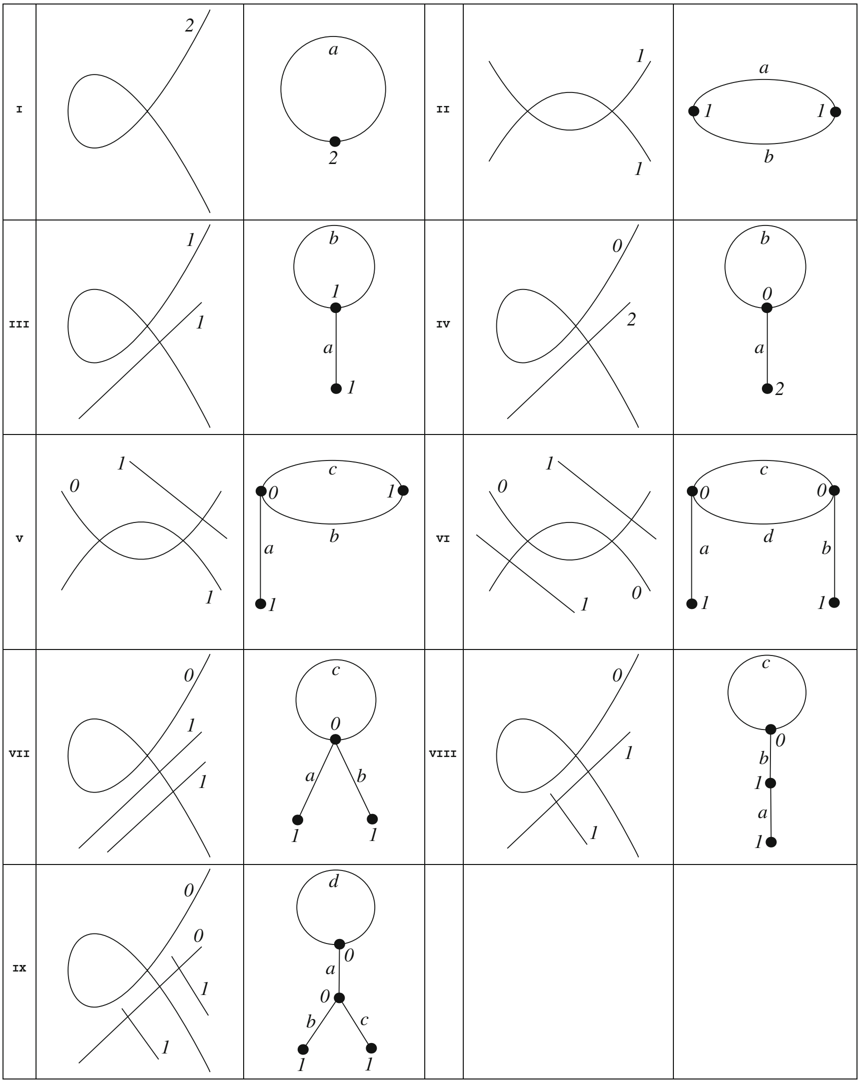

Fig. 3. Irreducible components of genus 3 curves and their dual graphs when the dual graphs have genus 1, i.e. when $\bar{g} = 3$ and $g = 1$

Based on these observations, Fig. 3 illustrates the possible pm-graphs satisfying these conditions.

We have $\tau(\Gamma) = \frac{\ell(\Gamma)}{4}$ when Γ is a tree metrized graph [2, Eq. 14.3]. Moreover, $\tau(\Gamma) = \frac{\ell(\Gamma)}{12}$ when Γ is a circle metrized graph [5, Corollary 2.17]. We use these facts and additive property of tau constant [5, page 15] to compute $\tau(\Gamma)$ for each pm-graphs listed in Fig. 3. We again use Equation (3) to compute $\theta(\Gamma)$. Then we compute $\varphi(\Gamma)$, $\lambda(\Gamma)$ and $\epsilon(\Gamma)$ by using Theorem 2.3. The invariants $\delta_0(\Gamma)$ and $\delta_1(\Gamma)$ are determined by using their definitions and by considering the topology of

Table 2. Pm-graph invariants when $g(\Gamma) = 1$, part 1

	$\ell(\Gamma)$	$\delta_0(\Gamma)$	$\delta_1(\Gamma)$	$\tau(\Gamma)$	$\theta(\Gamma)$
I	a	a	0	$\frac{\ell(\Gamma)}{12}$	0
II	$a+b$	$a+b$	0	$\frac{\ell(\Gamma)}{12}$	$\frac{8ab}{a+b}$
III	$a+b$	b	a	$\frac{\ell(\Gamma)}{12}+\frac{a}{6}$	$6a$
IV	$a+b$	b	a	$\frac{\ell(\Gamma)}{12}+\frac{a}{6}$	$6a$
V	$a+b+c$	$b+c$	a	$\frac{\ell(\Gamma)}{12}+\frac{a}{6}$	$6a+\frac{8bc}{b+c}$
VI	$a+b+c+d$	$d+c$	$a+b$	$\frac{\ell(\Gamma)}{12}+\frac{a+b}{6}$	$6(a+b)+\frac{8cd}{c+d}$
VII	$a+b+c$	c	$a+b$	$\frac{\ell(\Gamma)}{12}+\frac{a+b}{6}$	$6(a+b)$
VIII	$a+b+c$	c	$a+b$	$\frac{\ell(\Gamma)}{12}+\frac{a+b}{6}$	$6(a+b)$
IX	$a+b+c+d$	d	$a+b+c$	$\frac{\ell(\Gamma)}{12}+\frac{a+b+c}{6}$	$6(a+b+c)$

Table 3. Pm-graph invariants when $g(\Gamma) = 1$, part 2

	$\varphi(\Gamma)$	$\lambda(\Gamma)$	$\epsilon(\Gamma)$
I	$\frac{\ell(\Gamma)}{9}$	$\frac{3\ell(\Gamma)}{28}$	$\frac{2\ell(\Gamma)}{9}$
II	$\frac{\ell(\Gamma)}{9}+\frac{2ab}{3(a+b)}$	$\frac{3\ell(\Gamma)}{28}+\frac{ab}{7(a+b)}$	$\frac{2\ell(\Gamma)}{9}+\frac{4ab}{3(a+b)}$
III	$\frac{\ell(\Gamma)}{9}+\frac{11a}{9}$	$\frac{3\ell(\Gamma)}{28}+\frac{5a}{28}$	$\frac{2\ell(\Gamma)}{9}+\frac{13a}{9}$
IV	$\frac{\ell(\Gamma)}{9}+\frac{11a}{9}$	$\frac{3\ell(\Gamma)}{28}+\frac{5a}{28}$	$\frac{2\ell(\Gamma)}{9}+\frac{13a}{9}$
V	$\frac{\ell(\Gamma)}{9}+\frac{6bc+11a(b+c)}{9(b+c)}$	$\frac{3\ell(\Gamma)}{28}+\frac{4bc+5a(b+c)}{28(b+c)}$	$\frac{2\ell(\Gamma)}{9}+\frac{12bc+13a(b+c)}{9(b+c)}$
VI	$\frac{\ell(\Gamma)}{9}+\frac{6cd+11(a+b)(c+d)}{9(c+d)}$	$\frac{3\ell(\Gamma)}{28}+\frac{4cd+5(a+b)(c+d)}{28(c+d)}$	$\frac{2\ell(\Gamma)}{9}+\frac{12cd+13(a+b)(c+d)}{9(c+d)}$
VII	$\frac{\ell(\Gamma)}{9}+\frac{11(a+b)}{9}$	$\frac{3\ell(\Gamma)}{28}+\frac{5(a+b)}{28}$	$\frac{2\ell(\Gamma)}{9}+\frac{13(a+b))}{9}$
VIII	$\frac{\ell(\Gamma)}{9}+\frac{11(a+b)}{9}$	$\frac{3\ell(\Gamma)}{28}+\frac{5(a+b)}{28}$	$\frac{2\ell(\Gamma)}{9}+\frac{13(a+b)}{9}$
IX	$\frac{\ell(\Gamma)}{9}+\frac{11(a+b+c)}{9}$	$\frac{3\ell(\Gamma)}{28}+\frac{5(a+b+c)}{28}$	$\frac{2\ell(\Gamma)}{9}+\frac{13(a+b+c)}{9}$

Γ. The results are given in Tables 2 and 3. As can be seen from Table 3, we have $\varphi(\Gamma) \geq \frac{1}{9}\ell(\Gamma)$, $\lambda(\Gamma) \geq \frac{3\ell(\Gamma)}{28}$, $\epsilon(\Gamma) \geq \frac{2\ell(\Gamma)}{9}$, and these lower bounds are attained by the pm-graph given in part *I* of Fig. 3.

5. The case $g(\Gamma) = 2$

In this section, we consider pm-graphs $(\Gamma, \mathbf{q})$ with $g = 2$ and $\bar{g} = 3$. Using Equation (2) we see that $\mathbf{q}(p) = 1$ for only one vertex $p \in V(\Gamma)$ and that $\mathbf{q}(p) = 0$ for all the remaining vertices. By Remark 2.2 again, such a pm-graph can have at most one vertex with valence exactly 1. Moreover, we have $e = v + 1$. Based on these observations, Fig. 4 illustrates all the pm-graphs satisfying these conditions.

A metrized graph with two vertices and m multiple edges connecting these two vertices is called m-banana. For such Γ we know how to compute $\tau(\Gamma)$ [5, Proposition 8.3]. Then using tau formulas for tree and circle metrized graphs along

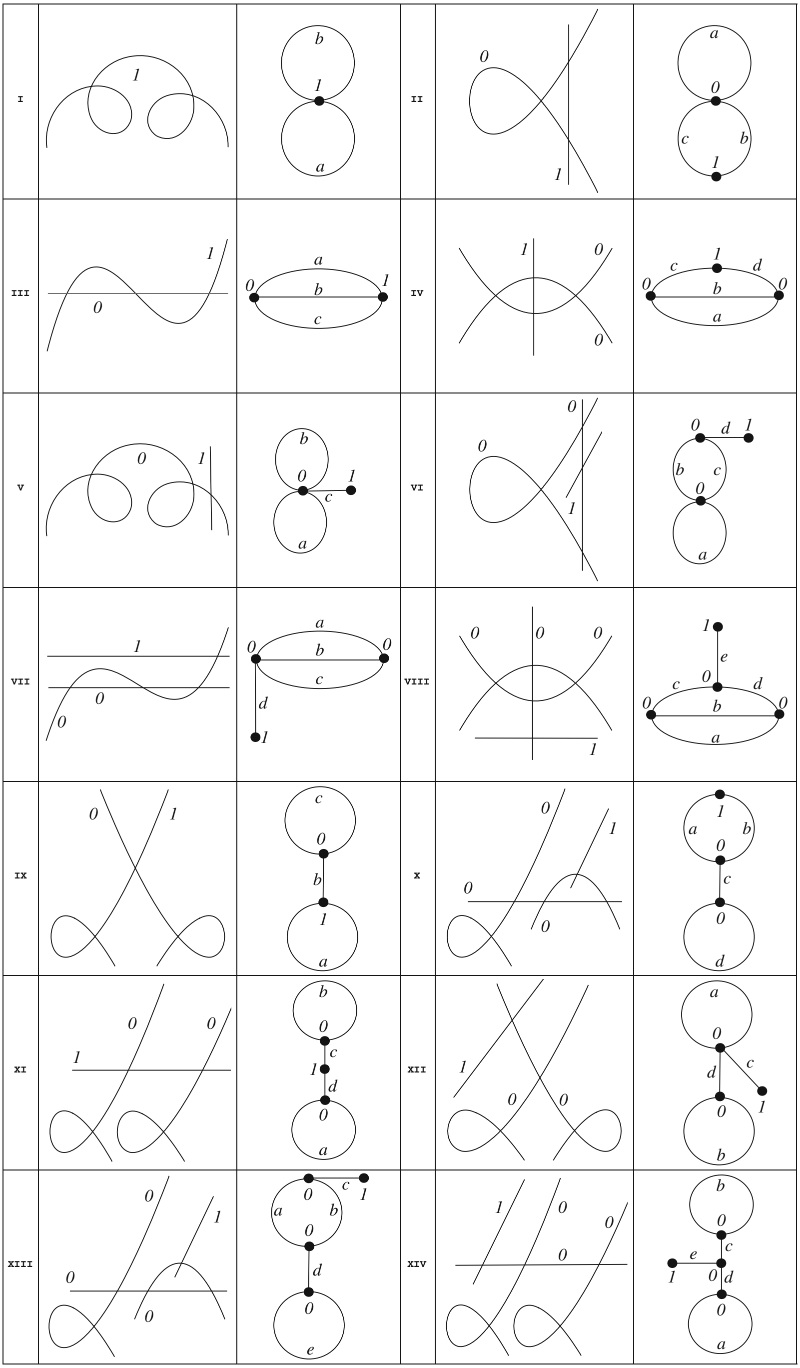

Fig. 4. Irreducible components of genus 3 curves and their dual graphs when the dual graphs have genus 2, i.e. when $\bar{g} = 3$ and $g = 2$

Table 4. Pm-graph invariants when $g(\Gamma)=2$, part 1. We have $\delta_0(\Gamma)=\ell(\Gamma)-\delta_1(\Gamma)$, and $\delta_i(\Gamma)=0$ for all $i\geq 2$

	$\ell(\Gamma)$	$\delta_1(\Gamma)$	$\tau(\Gamma)$	$\theta(\Gamma)$
I	$a+b$	0	$\frac{\ell(\Gamma)}{12}$	0
II	$a+b+c$	0	$\frac{\ell(\Gamma)}{12}$	$\frac{8bc}{b+c}$
III	$a+b+c$	0	$\frac{\ell(\Gamma)}{12}-\frac{abc}{6(ab+ac+bc)}$	$\frac{6abc}{ab+ac+bc}$
IV	$a+b+c+d$	0	$\frac{\ell(\Gamma)}{12}-\frac{ab(c+d)}{6(ab+(a+b)(c+d))}$	$\frac{6ab(c+d)+8(a+b)cd}{ab+(a+b)(c+d)}$
V	$a+b+c$	c	$\frac{\ell(\Gamma)}{12}+\frac{c}{6}$	$6c$
VI	$a+b+c+d$	d	$\frac{\ell(\Gamma)}{12}+\frac{d}{6}$	$6d+\frac{8bc}{b+c}$
VII	$a+b+c+d$	d	$\frac{\ell(\Gamma)}{12}+\frac{d}{6}-\frac{abc}{6(ab+ac+bc)}$	$6d+\frac{6abc}{ab+ac+bc}$
$VIII$	$a+b+c+d+e$	e	$\frac{\ell(\Gamma)}{12}+\frac{e}{6}-\frac{ab(c+d)}{6(ab+(a+b)(c+d))}$	$6e+\frac{6ab(c+d)+8(a+b)cd}{ab+(a+b)(c+d)}$
IX	$a+b+c$	b	$\frac{\ell(\Gamma)}{12}+\frac{b}{6}$	$6b$
X	$a+b+c+d$	c	$\frac{\ell(\Gamma)}{12}+\frac{c}{6}$	$6c+\frac{8ab}{a+b}$
XI	$a+b+c+d$	$c+d$	$\frac{\ell(\Gamma)}{12}+\frac{c+d}{6}$	$6(c+d)$
XII	$a+b+c+d$	$c+d$	$\frac{\ell(\Gamma)}{12}+\frac{c+d}{6}$	$6(c+d)$
$XIII$	$a+b+c+d+e$	$d+c$	$\frac{\ell(\Gamma)}{12}+\frac{c+d}{6}$	$6(c+d)+\frac{8ab}{a+b}$
XIV	$a+b+c+d+e$	$c+d+e$	$\frac{\ell(\Gamma)}{12}+\frac{c+d+e}{6}$	$6(c+d+e)$

Table 5. Pm-graph invariants when $g(\Gamma)=2$, part 2

	$\varphi(\Gamma)$	$\lambda(\Gamma)$	$\epsilon(\Gamma)$
I	$\frac{\ell(\Gamma)}{9}$	$\frac{3\ell(\Gamma)}{28}$	$\frac{2\ell(\Gamma)}{9}$
II	$\frac{\ell(\Gamma)}{9}+\frac{2bc}{3(b+c)}$	$\frac{3\ell(\Gamma)}{28}+\frac{bc}{7(b+c)}$	$\frac{2\ell(\Gamma)}{9}+\frac{4bc}{3(b+c)}$
III	$\frac{\ell(\Gamma)}{9}-\frac{2abc}{9(ab+ac+bc)}$	$\frac{3\ell(\Gamma)}{28}+\frac{abc}{28(ab+ac+bc)}$	$\frac{2\ell(\Gamma)}{9}+\frac{5abc}{9(ab+ac+bc)}$
IV	$\frac{\ell(\Gamma)}{9}+\frac{6cd(a+b)-2ab(c+d)}{9(ab+(a+b)(c+d))}$	$\frac{3\ell(\Gamma)}{28}+\frac{4cd(a+b)+ab(c+d)}{28(ab+(a+b)(c+d))}$	$\frac{2\ell(\Gamma)}{9}+\frac{12cd(a+b)+5ab(c+d)}{9(ab+(a+b)(c+d))}$
V	$\frac{\ell(\Gamma)}{9}+\frac{11c}{9}$	$\frac{3\ell(\Gamma)}{28}+\frac{5c}{28}$	$\frac{2\ell(\Gamma)}{9}+\frac{13c}{9}$
VI	$\frac{\ell(\Gamma)}{9}+\frac{6bc+11d(b+c)}{9(b+c)}$	$\frac{3\ell(\Gamma)}{28}+\frac{4bc+5d(b+c)}{28(b+c)}$	$\frac{2\ell(\Gamma)}{9}+\frac{12bc+13d(b+c)}{9(b+c)}$
VII	$\frac{\ell(\Gamma)}{9}+\frac{11d}{9}-\frac{2abc}{9(ab+ac+bc)}$	$\frac{3\ell(\Gamma)}{28}+\frac{5d}{28}+\frac{abc}{28(ab+ac+bc)}$	$\frac{2\ell(\Gamma)}{9}+\frac{13d}{9}+\frac{5abc}{9(ab+ac+bc)}$
$VIII$	$\frac{\ell(\Gamma)+11e}{9}+\frac{6cd(a+b)-2ab(c+d)}{9(ab+(a+b)(c+d))}$	$\frac{3\ell(\Gamma)+5e}{28}+\frac{4cd(a+b)+ab(c+d)}{28(ab+(a+b)(c+d))}$	$\frac{2\ell(\Gamma)+13e}{9}+\frac{12cd(a+b)+5ab(c+d)}{9(ab+(a+b)(c+d))}$
IX	$\frac{\ell(\Gamma)}{9}+\frac{11b}{9}$	$\frac{3\ell(\Gamma)}{28}+\frac{5b}{28}$	$\frac{2\ell(\Gamma)}{9}+\frac{13b}{9}$
X	$\frac{\ell(\Gamma)}{9}+\frac{6ab+11c(a+b)}{9(a+b)}$	$\frac{3\ell(\Gamma)}{28}+\frac{4ab+5c(a+b)}{28(a+b)}$	$\frac{2\ell(\Gamma)}{9}+\frac{12ab+13c(a+b)}{9(a+b)}$
XI	$\frac{\ell(\Gamma)}{9}+\frac{11(c+d)}{9}$	$\frac{3\ell(\Gamma)}{28}+\frac{5(c+d)}{28}$	$\frac{2\ell(\Gamma)}{9}+\frac{13(c+d)}{9}$
XII	$\frac{\ell(\Gamma)}{9}+\frac{11(c+d)}{9}$	$\frac{3\ell(\Gamma)}{28}+\frac{5(c+d)}{28}$	$\frac{2\ell(\Gamma)}{9}+\frac{13(c+d)}{9}$
$XIII$	$\frac{\ell(\Gamma)}{9}+\frac{6ab+11(a+b)(c+d)}{9(a+b)}$	$\frac{3\ell(\Gamma)}{28}+\frac{4ab+5(a+b)(c+d)}{28(a+b)}$	$\frac{2\ell(\Gamma)}{9}+\frac{12ab+13(a+b)(c+d)}{9(a+b)}$
XIV	$\frac{\ell(\Gamma)}{9}+\frac{11(c+d+e)}{9}$	$\frac{3\ell(\Gamma)}{28}+\frac{5(c+d+e)}{28}$	$\frac{2\ell(\Gamma)}{9}+\frac{13(c+d+e)}{9}$

with the additive property, one can compute $\tau(\Gamma)$ for each of the pm-graphs given in Fig. 4. Again we use its definition to compute $\theta(\Gamma)$. Then we compute the remaining invariants $\varphi(\Gamma)$, $\lambda(\Gamma)$ and $\epsilon(\Gamma)$ by using Theorem 2.3. The invariants $\delta_0(\Gamma)$ and $\delta_1(\Gamma)$ are determined by using their definitions and by considering the topology of Γ. The results are given in Tables 4 and 5.

As can be seen from the values of $\varphi(\Gamma)$, we have $\varphi(\Gamma) \geq \frac{1}{9}\ell(\Gamma)$ if Γ is one of the pm-graphs given in parts I, II, V, VI, IX, X, XI, XII, $XIII$, XIV.

For the pm-graph Γ of type III, we see that $\frac{a+b+c}{3} - \frac{3}{1/a+1/b+1/c} \geq 0$ by Arithmetic-Harmonic Mean inequality. Note that $\ell(\Gamma) = a+b+c$. Therefore, we have $\varphi(\Gamma) \geq \frac{1}{9}\ell(\Gamma) - \frac{2}{9}\frac{1}{9}\ell(\Gamma) = \frac{7}{81}\ell(\Gamma)$, which is the sharp lower bound for this type of pm-graphs, because $\varphi(\Gamma) = \frac{7}{81}\ell(\Gamma)$ whenever $a = b = c$.

Using the same inequality $\frac{a+b+c}{3} - \frac{3}{1/a+1/b+1/c} \geq 0$, we see that $\varphi(\Gamma) \geq \frac{7}{81}\ell(\Gamma)$ for the pm-graph of type VII.

Similarly, if we use Arithmetic-Harmonic Mean inequality for a, b and $c+d$, we again obtain that $\varphi(\Gamma) \geq \frac{7}{81}\ell(\Gamma)$ for the pm-graphs as in type IV and $VIII$.

In any case, we have the sharp lower bound $\varphi(\Gamma) \geq \frac{7}{81}\ell(\Gamma)$ whenever $g = 2$.

Using the results from Table 5 we have, as in the previous section, $\lambda(\Gamma) \geq \frac{3\ell(\Gamma)}{28}$ and $\epsilon(\Gamma) \geq \frac{2\ell(\Gamma)}{9}$, and these lower bounds are attained by the pm-graph given in part I of Fig. 4.

6. The case $g(\Gamma) = 3$

In this section, we consider pm-graphs $(\Gamma, \mathbf{q})$ with $g(\Gamma) = 3$ and $g = 3$. In this case, Equation (2) implies $\mathbf{q}(p) = 0$ for each vertex $p \in V(\Gamma)$. That is, $(\Gamma, \mathbf{q})$ is a simple pm-graph. Using this observation and Remark 2.2, we note that $\upsilon(p) \geq 2$ for each $p \in V(\Gamma)$. Moreover, using Remark 2.4 we can assume that $\upsilon(p) \geq 3$ for each $p \in V(\Gamma)$ for this section. By basic graph theory, this implies $e \geq \frac{3}{2}v$. On the other hand, we have $e = v + 2$ since $g = 3$. Therefore, we conclude that $1 \leq v \leq 4$ for the simple pm-graphs we can have. Based on these observations, Fig. 5 illustrates all the pm-graphs satisfying these conditions.

We compute $\tau(\Gamma)$ by using similar techniques as in the previous section except for the pm-graphs in parts $VIII$, $XIII$ and XIV of Fig. 5. The simple pm-graph in part XIV is a tetrahedral graph for which we have computed its tau constant in [5, Example 8.4]. We can compute the tau constant for the simple pm-graphs in parts $VIII$ and $XIII$ by using the techniques developed in [5], such as [5, Corollaries 5.3 and 7.4, Propositions 4.6 and 4.5].

As in the previous sections, we compute $\theta(\Gamma)$ by using its definition and determining the resistance values between any two vertices of the pm-graph. Note that computing the resistance matrix of the corresponding graph will also help, as it is done in [10, Example III].

Once the values of $\tau(\Gamma)$ and $\theta(\Gamma)$ are obtained, we compute $\varphi(\Gamma)$, $\lambda(\Gamma)$ and $\epsilon(\Gamma)$ by using Theorem 2.3. As in the previous sections, the topology of Γ is the main factor effecting the invariants $\delta_0(\Gamma)$ and $\delta_1(\Gamma)$.

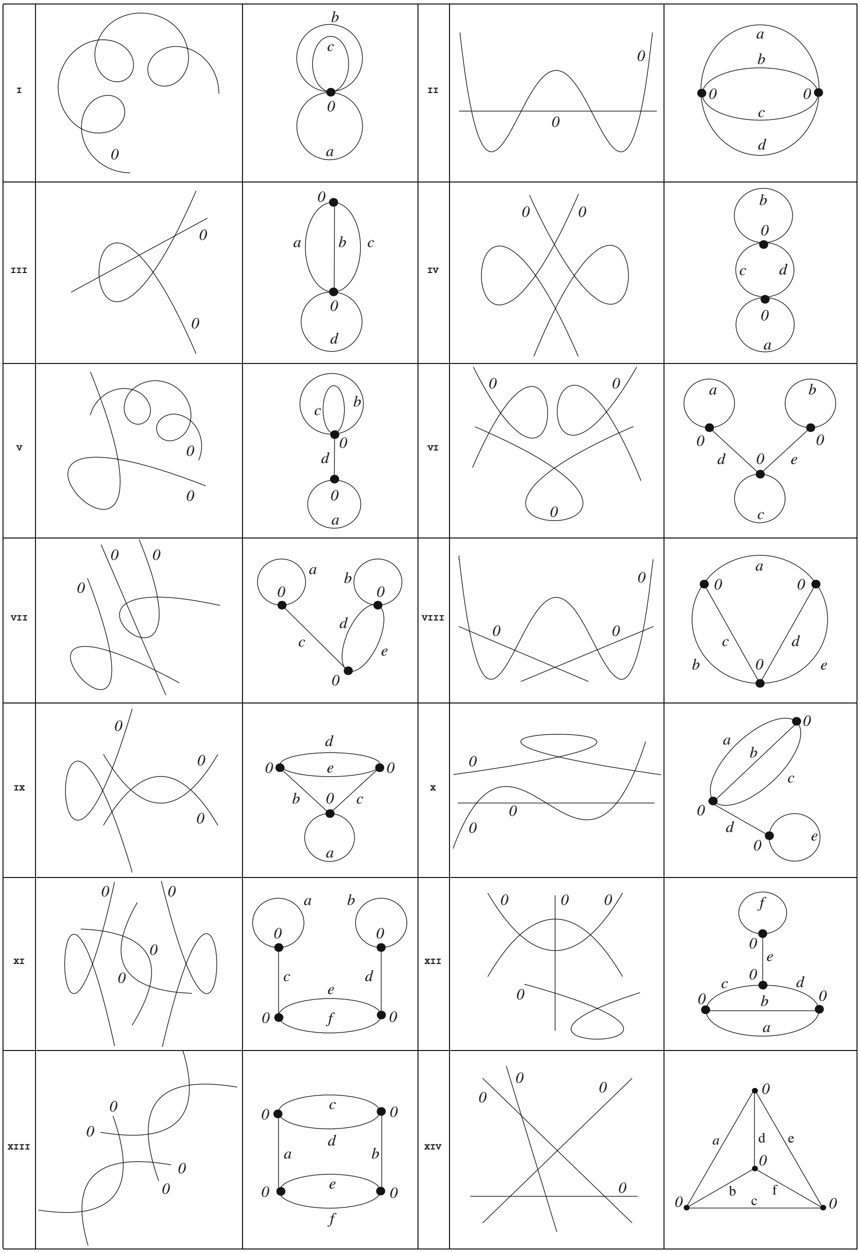

Fig. 5. Irreducible components of genus 3 curves and their dual graphs when the dual graphs have genus 3, i.e. when $\bar{g} = g = 3$

Table 6. Values of $\ell(\Gamma)$, $\delta_1(\Gamma)$ and $\tau(\Gamma)$ for pm-graphs with $g(\Gamma)=3$ and $\bar{g}=3$. We have $\delta_0(\Gamma)=\ell(\Gamma)-\delta_1(\Gamma)$, and $\delta_i(\Gamma)=0$ for all $i\geq 2$

	$\ell(\Gamma)$	$\delta_1(\Gamma)$	$\tau(\Gamma)$
I	$a+b+c$	0	$\frac{\ell(\Gamma)}{12}$
II	$a+b+c+d$	0	$\frac{\ell(\Gamma)}{12}-\frac{abcd}{3(bcd+a(cd+b(c+d)))}$
III	$a+b+c+d$	0	$\frac{\ell(\Gamma)}{12}-\frac{abc}{6(ab+ac+bc)}$
IV	$a+b+c+d$	0	$\frac{\ell(\Gamma)}{12}$
V	$a+b+c+d$	d	$\frac{\ell(\Gamma)}{12}+\frac{d}{6}$
VI	$a+b+c+d+e$	$d+e$	$\frac{\ell(\Gamma)}{12}+\frac{d+e}{6}$
VII	$a+b+c+d+e$	c	$\frac{\ell(\Gamma)}{12}+\frac{c}{6}$
$VIII$	$a+b+c+d+e$	0	$\frac{\ell(\Gamma)}{12}-\frac{abcd+abce+abde+acde+2bcde}{6(abd+acd+bcd+abe+ace+bce+bde+cde)}$
IX	$a+b+c+d+e$	0	$\frac{\ell(\Gamma)}{12}+\frac{b}{6}$
X	$a+b+c+d+e$	d	$\frac{\ell(\Gamma)}{12}+\frac{d}{6}-\frac{abc}{6(ab+ac+bc)}$
XI	$a+b+c+d+e+f$	$c+d$	$\frac{\ell(\Gamma)}{12}+\frac{c+d}{6}$
XII	$a+b+c+d+e+f$	e	$\frac{\ell(\Gamma)}{12}+\frac{e}{6}-\frac{ab(c+d)}{6(ab+(a+b)(c+d))}$

Table 7. Values of $\theta(\Gamma)$ and $\varphi(\Gamma)$ for pm-graphs with $g(\Gamma)=3$ and $\bar{g}=3$

	$\theta(\Gamma)$	$\varphi(\Gamma)$
I	0	$\frac{\ell(\Gamma)}{9}$
II	$\frac{8abcd}{bcd+a(cd+b(c+d))}$	$\frac{\ell(\Gamma)}{9}-\frac{7abcd}{9(bcd+a(cd+b(c+d)))}$
III	$\frac{6abc}{ab+ac+bc}$	$\frac{\ell(\Gamma)}{9}-\frac{2abc}{9(bc+a(b+c))}$
IV	$\frac{8cd}{c+d}$	$\frac{\ell(\Gamma)}{9}+\frac{2cd}{3(c+d)}$
V	$6d$	$\frac{\ell(\Gamma)}{9}+\frac{11d}{9}$
VI	$6(d+e)$	$\frac{\ell(\Gamma)}{9}+\frac{11(d+e)}{9}$
VII	$6c+\frac{8de}{d+e}$	$\frac{\ell(\Gamma)}{9}+\frac{11c}{9}+\frac{2de}{3(d+e)}$
$VIII$	$\frac{6abcd+6abce+6abde+6acde+8bcde}{abd+acd+bcd+abe+ace+bce+bde+cde}$	$\frac{\ell(\Gamma)}{9}-\frac{7bcde+2a(cde+b(de+c(d+e)))}{9(cde+a(b+c)(d+e)+b(de+c(d+e)))}$
IX	$\frac{8bcd+8bce+6bde+6cde}{bd+cd+be+ce+de}$	$\frac{\ell(\Gamma)}{9}+\frac{-2(b+c)de+6bc(d+e)}{9(de+(b+c)(d+e))}$
X	$6d+\frac{6abc}{ab+ac+bc}$	$\frac{\ell(\Gamma)}{9}+\frac{11d}{9}-\frac{2abc}{9(bc+a(b+c))}$
XI	$6(c+d)+\frac{8ef}{e+f}$	$\frac{\ell(\Gamma)}{9}+\frac{11(c+d)}{9}+\frac{2ef}{3(e+f)}$
XII	$6e+\frac{6abc+6abd+8acd+8bcd}{ab+(a+b)(c+d)}$	$\frac{\ell(\Gamma)}{9}+\frac{11e}{9}+\frac{6(a+b)cd-2ab(c+d)}{9(ab+(a+b)(c+d))}$

The results for pm-graphs of types I-XII in Fig. 5 are given in Tables 6, 7 and 8. Since the values of the invariants are lengthy for the remaining pm-graphs, types $XIII$ and XIV, we state them separately in this section.

It is clear from Table 8 and the results for pm-graphs of types $XIII$ and XIV that $\lambda(\Gamma)\geq\frac{3}{28}\ell(\Gamma)$ and $\epsilon(\Gamma)\geq\frac{2}{9}\ell(\Gamma)$ for all pm-graphs in Fig. 5, and this lower bounds are attained for the pm-graph in type I.

Clearly, Table 7 shows that $\varphi(\Gamma)\geq\frac{1}{9}\ell(\Gamma)$ for pm-graphs of type I, IV, V, VI, VII, XI.

We use Arithmetic-Harmonic Mean inequality for a, b, c, i.e., $\frac{a+b+c}{3}-\frac{3}{1/a+1/b+1/c}\geq 0$, to derive $\varphi(\Gamma)\geq\frac{7}{81}\ell(\Gamma)$ for the pm-graphs of types III

Table 8. Values of $\lambda(\Gamma)$ and $\epsilon(\Gamma)$ for pm-graphs with $g(\Gamma) = 3$ and $\bar{g} = 3$

	$\lambda(\Gamma)$	$\epsilon(\Gamma)$
I	$\frac{3\ell(\Gamma)}{28}$	$\frac{2\ell(\Gamma)}{9}$
II	$\frac{3\ell(\Gamma)}{28}$	$\frac{2\ell(\Gamma)}{9} + \frac{4abcd}{9(bcd+a(cd+b(c+d)))}$
III	$\frac{3\ell(\Gamma)}{28} + \frac{abc}{28(bc+a(b+c))}$	$\frac{2\ell(\Gamma)}{9} + \frac{5abc}{9(bc+a(b+c))}$
IV	$\frac{3\ell(\Gamma)}{28} + \frac{cd}{7(c+d)}$	$\frac{2\ell(\Gamma)}{9} + \frac{4cd}{3(c+d)}$
V	$\frac{3\ell(\Gamma)}{28} + \frac{5d}{28}$	$\frac{2\ell(\Gamma)}{9} + \frac{13d}{9}$
VI	$\frac{3\ell(\Gamma)}{28} + \frac{5(d+e)}{28}$	$\frac{2\ell(\Gamma)}{9} + \frac{13(d+e)}{9}$
VII	$\frac{3\ell(\Gamma)}{28} + \frac{5c}{28} + \frac{de}{7(d+e)}$	$\frac{2\ell(\Gamma)}{9} + \frac{13c}{9} + \frac{4de}{3(d+e)}$
$VIII$	$\frac{3\ell(\Gamma)}{28} + \frac{a((c+b)de+bc(d+e))}{28(cde+a(b+c)(d+e)+b(de+c(d+e)))}$	$\frac{2\ell(\Gamma)}{9} + \frac{4bcde+5a(cde+b(de+c(d+e)))}{9(cde+a(b+c)(d+e)+b(de+c(d+e)))}$
IX	$\frac{3\ell(\Gamma)}{28} + \frac{(b+c)de+4bc(d+e)}{28(de+(b+c)(d+e))}$	$\frac{2\ell(\Gamma)}{9} + \frac{5(b+c)de+12bc(d+e)}{9(de+(b+c)(d+e))}$
X	$\frac{3\ell(\Gamma)}{28} + \frac{5d}{28} + \frac{abc}{28(bc+a(b+c))}$	$\frac{2\ell(\Gamma)}{9} + \frac{13d}{9} + \frac{5abc}{9(bc+a(b+c))}$
XI	$\frac{3\ell(\Gamma)}{28} + \frac{5(c+d)}{28} + \frac{ef}{7(e+f)}$	$\frac{2\ell(\Gamma)}{9} + \frac{13(c+d)}{9} + \frac{4ef}{3(e+f)}$
XII	$\frac{3\ell(\Gamma)}{28} + \frac{5e}{28} + \frac{4(a+b)cd+ab(c+d)}{28(ab+(a+b)(c+d))}$	$\frac{2\ell(\Gamma)}{9} + \frac{13e}{9} + \frac{12(a+b)cd+5ab(c+d)}{9(ab+(a+b)(c+d))}$

and X. Using the Arithmetic-Harmonic Mean inequality for $b+c$, d, e gives $\varphi(\Gamma) \geq \frac{7}{81}\ell(\Gamma)$ for the pm-graph of type IX. Similarly, using the Arithmetic-Harmonic Mean inequality for a, b, $c+d$ gives $\varphi(\Gamma) \geq \frac{7}{81}\ell(\Gamma)$ for the pm-graph of type XII.

We note that $\varphi(\Gamma) = \frac{1}{16}\ell(\Gamma) + \frac{7}{36}(\frac{a+b+c+d}{4} - \frac{4}{1/a+1/b+1/c+1/d})$ for the pm-graph of type II. Therefore, we obtain $\varphi(\Gamma) \geq \frac{1}{16}\ell(\Gamma)$ by using the Arithmetic-Harmonic Mean inequality for a, b, c and d. In particular, $\varphi(\Gamma) = \frac{1}{16}\ell(\Gamma)$ when $a=b=c=d$.

Computations to find the lower bounds of $\varphi(\Gamma)$ requires more in-depth analysis for pm-graphs of types $VIII$, $XIII$ and XIV. Thus, we consider each of these pm-graphs separately.

Pm-graphs of type $VIII$:

Let $H = b^2cd+bc^2d+bcd^2+b^2ce+bc^2e+b^2de-12bcde+c^2de+bd^2e+cd^2e+bce^2+bde^2+cde^2$, $D = ace+abe+cbe+acd+abd+cbd+ced+bed$, $N = 14ace+3c^2e+14abe+3b^2e+3ce^2+3be^2+14acd+3c^2d+14abd+3b^2d+3cd^2+3bd^2$, $M = (c-b)^2e+(c-b)^2d+c(e-d)^2+b(e-d)^2$. Clearly, D, N and M are nonnegative. We note that $H \geq 0$ by applying Arithmetic-Harmonic Mean inequality for b, c, d and e.

Now we note that $\varphi(\Gamma) = \frac{1}{16}\ell(\Gamma) + \frac{a(N+11M)+14H}{288D}$, which implies that $\varphi(\Gamma) \geq \frac{1}{16}\ell(\Gamma)$. This is the sharp lower bound because $\varphi(\Gamma) = \frac{1}{16}\ell(\Gamma)$ whenever $a=0$ and $b=c=d=e$.

Pm-graphs of type $XIII$:

Let Γ be a pm-graph as illustrated in $XIII$ in Fig. 5. In this case, we have $\ell(\Gamma) = a+b+c+d+e+f$, $\delta_0(\Gamma) = \ell(\Gamma)$, and $\delta_i(\Gamma) = 0$ for all $i \geq 1$.

Moreover,

$$\tau(\Gamma) = \frac{\ell(\Gamma)}{12} - \frac{A+2C}{6D}, \qquad \theta(\Gamma) = \frac{6A+8B+8C}{D},$$
$$\varphi(\Gamma) = \frac{\ell(\Gamma)}{9} - \frac{2A-6B+7C}{9D}, \qquad \lambda(\Gamma) = \frac{3\ell(\Gamma)}{28} + \frac{A+4B}{28D},$$

and

$$\epsilon(\Gamma) = \frac{2}{9}\ell(\Gamma) + \frac{5A+12B+4C}{9D},$$

where $A = acde + bcde + acdf + bcdf + acef + bcef + adef + bdef$, $B = abce + abde + abcf + abdf$, $C = cdef$ and $D = (a+b)ce + (a+b)de + cde + (a+b)cf + (a+b)df + cdf + cef + def)$.

Let $H = c^2de + cd^2e + cde^2 + c^2df + cd^2f + c^2ef - 12cdef + d^2ef + ce^2f + de^2f + cdf^2 + cef^2 + def^2$, $N = 14(a+b)ce + 3c^2e + 14(a+b)de + 3d^2e + 3ce^2 + 3de^2 + 14(a+b)cf + 3c^2f + 14(a+b)df + 3d^2f + 3cf^2 + 3df^2$, $M = (c-d)^2e + (c-d)^2f + c(e-f)^2 + d(e-f)^2$. We see that D, N and M are nonnegative, and note that $H \geq 0$ by applying Arithmetic-Harmonic Mean inequality for c, d, e and f.

Now we note that $\varphi(\Gamma) = \frac{1}{16}\ell(\Gamma) + \frac{(a+b)(N+11M)+14H+192ab(c+d)(e+f)}{288D}$, which implies that $\varphi(\Gamma) \geq \frac{1}{16}\ell(\Gamma)$. This is the sharp lower bound because $\varphi(\Gamma) = \frac{1}{16}\ell(\Gamma)$ whenever $a = b = 0$ and $c = d = e = f$.

Pm-graphs of type XIV:

Let Γ be a pm-graph as illustrated in XIV in Fig. 5. In this case, we have $\ell(\Gamma) = a+b+c+d+e+f$, $\delta_0(\Gamma) = \ell(\Gamma)$, and $\delta_i(\Gamma) = 0$ for all $i \geq 1$. Moreover,

$$\tau(\Gamma) = \frac{\ell(\Gamma)}{12} - \frac{A+2B}{6C}, \qquad \theta(\Gamma) = \frac{6A+8B}{C},$$
$$\varphi(\Gamma) = \frac{\ell(\Gamma)}{9} - \frac{2A+7B}{9C}, \qquad \lambda(\Gamma) = \frac{3\ell(\Gamma)}{28} + \frac{A}{28C},$$
$$\epsilon(\Gamma) = \frac{2}{9}\ell(\Gamma) + \frac{5A+4B}{9C},$$

where $A = abcd + abce + abde + acde + abcf + abdf + bcdf + acef + bcef + adef + bdef + cdef$, $B = bcde + acdf + abef$, and $C = abd + acd + bcd + abe + ace + bce + bde + cde + abf + acf + bcf + adf + cdf + aef + bef + def$.

We first show that $\tau(\Gamma) \geq \frac{5}{96}\ell(\Gamma)$, where the equality holds whenever $a = b = c = d = e = f$. We have $\tau(\Gamma) = \frac{5}{96}\ell(\Gamma) + \frac{3M-7A-20B}{96C}$ where $M = a^2bd + a^2be + a^2bf + a^2cd + a^2ce + a^2cf + a^2df + a^2ef + ab^2d + ab^2e + ab^2f + abd^2 + abe^2 + abf^2 + ac^2d + ac^2e + ac^2f + acd^2 + ace^2 + acf^2 + ad^2f + adf^2 + ae^2f + aef^2 + b^2cd + b^2ce + b^2cf + b^2de + b^2ef + bc^2d + bc^2e + bc^2f + bcd^2 + bce^2 + bcf^2 + bd^2e + bde^2 + be^2f + bef^2 + c^2de + c^2df + cd^2e + cd^2f + cde^2 + cdf^2 + d^2ef + de^2f + def^2$.

Thus, we see that proving $S := 3M - 7A - 20B \geq 0$ gives $\tau(\Gamma) \geq \frac{5}{96}\ell(\Gamma)$. Now, we have the following tricky equality

$$\begin{aligned}
S = 2\Big[&be\left((a-f)^2+(d-c)^2\right)+cd\left((a-f)^2+(b-e)^2\right)\\
&+af\left((e-b)^2+(d-c)^2\right)\Big]+\frac{3}{2}\Big[bd\left((a-c)^2+(a-e)^2+(c-e)^2\right)\\
&+ce\left((b-a)^2+(d-a)^2+(b-d)^2\right)+ad\left((b-c)^2+(b-f)^2+(c-f)^2\right)\\
&+cf\left((a-b)^2+(a-d)^2+(d-b)^2\right)+bf\left((c-a)^2+(e-a)^2+(c-e)^2\right)\\
&+ae\left((b-c)^2+(f-b)^2+(f-c)^2\right)+ef\left((a-b)^2+(a-d)^2+(d-b)^2\right)\\
&+ab\left((d-e)^2+(d-f)^2+(f-e)^2\right)+ac\left((d-e)^2+(d-f)^2+(e-f)^2\right)\\
&+df\left((a-c)^2+(e-a)^2+(e-c)^2\right)\\
&+de\left((b-c)^2+(b-f)^2+(c-f)^2\right)+bc\left((d-e)^2+(d-f)^2+(e-f)^2\right)\Big]\\
&+\frac{1}{2}\Big[cd(a-b)^2+be(a-c)^2+af(b-c)^2+be(a-d)^2+af(b-d)^2\\
&\quad+cd(a-e)^2+af(c-e)^2+af(d-e)^2\\
&\quad+be\left((c-f)^2+(d-f)^2\right)+cd(b-f)^2+cd(e-f)^2\Big].
\end{aligned}$$

Thus, S is a sum of positive terms. This gives

$$3M - 7A - 20B \geq 0, \quad \text{and so} \quad \tau(\Gamma) \geq \frac{5}{96}\ell(\Gamma). \tag{5}$$

Now, we consider $\varphi(\Gamma)$. We note that $\varphi(\Gamma) = \frac{17}{288}\ell(\Gamma)$ whenever Γ has equal edge lengths, i.e., if $a = b = c = d = e = f$. Next, we show that this is the sharp lower bound for Γ and so for all pm-graphs of $\bar{g} = 3$.

Claim. $\varphi(\Gamma) \geq \frac{17}{288}\ell(\Gamma)$.

Proof of Claim. It is worth mentioning that we are unable to give a proof of this inequality neither by utilizing arithmetic harmonic mean inequalities partially or fully as in the previous cases nor by using any other well-known inequality in literature. Instead we found the following highly tricky and technical proof after spending extensive time on this problem.

Since $\varphi(\Gamma) = \frac{17}{288}\ell(\Gamma) + \frac{15D-19A-164B}{C}$, where A, B and C are as above and D is as follows:

$D = a^2bd + a^2be + a^2bf + a^2cd + a^2ce + a^2cf + a^2df + a^2ef + ab^2d + ab^2e + ab^2f + abd^2 + abe^2 + abf^2 + ac^2d + ac^2e + ac^2f + acd^2 + ace^2 + acf^2 + ad^2f + adf^2 + ae^2f + aef^2 + b^2cd + b^2ce + b^2cf + b^2de + b^2ef + bc^2d + bc^2e + bc^2f + bcd^2 + bce^2 + bcf^2 + bd^2e + bde^2 + be^2f + bef^2 + c^2de + c^2df + cd^2e + cd^2f + cde^2 + cdf^2 + d^2ef + de^2f + def^2$

Therefore, it is enough to show that the following inequality holds to prove the claim:

$$15D - 19A - 164B \geq 0 \tag{6}$$

The proof of this inequality consists of eight similar cases that depends on the comparison of the involved variables. The idea is to express $15D - 19A - 164B$ as sums squares and nonnegative terms. Lets denote this term by R, i.e., we set $R := 15D - 19A - 164B$. The number of cases can be reduced by considering the symmetries of the tetrahedral graph as we explain later, but we want to consider these eight cases explicitly.

Case I: Suppose $a \geq f$, $b \geq e$ and $c \geq d$:

We have

$$\begin{aligned} R = {} & 2\big[cd(b+e-a-f)^2 + be(c+d-a-f)^2 + af(c+d-b-e)^2\big] \\ & +13\big[be(a-c+d-f)^2 + cd(a-b+e-f)^2 + af(-b+c-d+e)^2\big] \\ & +15\big[de(b-c)^2 + bd(c-e)^2 + df(c-a)^2 + da(c-f)^2 \\ & \quad +ab(e-f)^2 + ac(d-f)^2 + ae(b-f)^2 + bc(d-e)^2 \\ & \quad +bf(a-e)^2 + ce(b-d)^2 + cf(a-d)^2 + ef(a-b)^2\big] \\ & +11\big[cd(a-f)(b-e) + be(a-f)(c-d) + af(b-e)(c-d)\big] + 15T_1, \end{aligned}$$

where $T_1 = a^2bd + a^2ce + ab^2d + abd^2 - 2abde - 2abdf + ac^2e - 2acde + ace^2 - 2acef + b^2cf + bc^2f - 2bcdf - 2bcef + bcf^2 + d^2ef + de^2f + def^2$.

By the assumptions in this case, we have $a - f \geq 0$, $b - e \geq 0$ and $c - d \geq 0$. Therefore, to prove $R \geq 0$, it will be enough to show $T_1 \geq 0$. Again by the assumptions, we can write $a = f + k$, $b = e + m$ and $c = d + n$ for some nonnegative real numbers k, m and n. Now substituting these into T_1 gives

$T_1 = d^2km + 2dek^2 + 2dfm^2 + dk^2m + dkm^2 + e^2kn + 2efn^2 + ek^2n + ekn^2 + f^2mn + fm^2n + fmn^2$, which clearly shows that $T_1 \geq 0$. Hence, $R \geq 0$ in this case.

Case II: Suppose $a \geq f$, $b \geq e$ and $d \geq c$:

We have

$$\begin{aligned} R = {} & 2\big[af(-b+c+d-e)^2 + be(-a+c+d-f)^2 + cd(-a+b+e-f)^2\big] \\ & +13\big[be(a+c-d-f)^2 + cd(a-b+e-f)^2 + af(-b-c+d+e)^2\big] \\ & +15\big[ae(b-f)^2 + ab(e-f)^2 + bf(a-e)^2 + ef(a-b)^2 + df(c-a)^2 \\ & \quad +ad(c-f)^2 + ac(d-f)^2 + cf(a-d)^2 + bd(c-e)^2 \\ & \quad +bc(d-e)^2 + ce(b-d)^2 + de(b-c)^2\big] \\ & +11\big[cd(a-f)(b-e) + be(a-f)(d-c) + af(b-e)(d-c)\big] + 15T_2, \end{aligned}$$

where $T_2 = a^2bd + a^2ce + ab^2d - 2abce - 2abcf + abd^2 + ac^2e - 2acde + ace^2 - 2adef + b^2cf + bc^2f - 2bcdf + bcf^2 - 2bdef + d^2ef + de^2f + def^2$.

By the assumptions in this case, we have $a - f \geq 0$, $b - e \geq 0$ and $d - c \geq 0$. Therefore, to prove $R \geq 0$, it will be enough to show $T_2 \geq 0$. Again by the assumptions, we can write $a = f + k$, $b = e + m$ and $d = c + n$ for some nonnegative real numbers k, m and n. Now substituting these into T_2 gives $T_2 = c^2km + 2cek^2 + 2cfm^2 + ck^2m + ckm^2 + 2ckmn + e^2kn + 2efn^2 + ek^2n + 2ekmn + ekn^2 + f^2mn + 2fkmn + fm^2n + fmn^2 + k^2mn + km^2n + kmn^2$. This shows that $T_2 \geq 0$. Hence, $R \geq 0$ in this case.

Case III: Suppose $a \geq f$, $e \geq b$ and $c \geq d$:

We have

$$\begin{aligned} R = {} & 2\big[af(-b+c+d-e)^2 + be(-a+c+d-f)^2 + cd(-a+b+e-f)^2\big] \\ & +13\big[cd(a-f+b-e)^2 + be(a-f-c+d)^2 + af(c-d+b-e)^2\big] \\ & +15\big[ae(b-f)^2 + ab(e-f)^2 + bf(a-e)^2 + ef(a-b)^2 + df(c-a)^2 \\ & \quad +ad(c-f)^2 + ac(d-f)^2 + cf(a-d)^2 + bd(c-e)^2 \\ & \quad +bc(d-e)^2 + ce(b-d)^2 + de(b-c)^2\big] \\ & +11\big[cd(a-f)(e-b) + be(a-f)(c-d) + af(e-b)(c-d)\big] + 15T_3, \end{aligned}$$

where $T_3 = a^2bd + a^2ce + ab^2d - 2abcd - 2abcf + abd^2 - 2abde + ac^2e + ace^2 - 2adef + b^2cf + bc^2f - 2bcef + bcf^2 - 2cdef + d^2ef + de^2f + def^2$.

By the assumptions in this case, we have $a - f \geq 0$, $e - b \geq 0$ and $c - d \geq 0$. Therefore, to prove $R \geq 0$, it will be enough to show $T_3 \geq 0$. Again by the assumptions, we can write $a = f + k$, $e = b + m$ and $c = d + n$ for some nonnegative real numbers k, m and n. Now substituting these into T_3 gives $T_3 = b^2kn + 2bdk^2 + 2bfn^2 + bk^2n + 2bkmn + bkn^2 + d^2km + 2dfm^2 + dk^2m + dkm^2 + 2dkmn + f^2mn + 2fkmn + fm^2n + fmn^2 + k^2mn + km^2n + kmn^2$, which shows that $T_3 \geq 0$. Hence, $R \geq 0$ in this case.

Case IV: Suppose $a \geq f$, $e \geq b$ and $d \geq c$:

We have

$$\begin{aligned} R = {} & 2\big[af(-b+c+d-e)^2 + be(-a+c+d-f)^2 + cd(-a+b+e-f)^2\big] \\ & +13\big[be(a+c-d-f)^2 + cd(a+b-e-f)^2 + af(b-c+d-e)^2\big] \\ & +15\big[ae(b-f)^2 + ab(e-f)^2 + bf(a-e)^2 + ef(a-b)^2 + df(c-a)^2 \\ & \quad +ad(c-f)^2 + ac(d-f)^2 + cf(a-d)^2 + bd(c-e)^2 \\ & \quad +bc(d-e)^2 + ce(b-d)^2 + de(b-c)^2\big] \\ & +11\big[cd(a-f)(e-b) + be(a-f)(d-c) + af(e-b)(d-c)\big] + 15T_4, \end{aligned}$$

where $T_4 = a^2bd + a^2ce + ab^2d - 2abcd - 2abce + abd^2 - 2abdf + ac^2e + ace^2 - 2acef + b^2cf + bc^2f + bcf^2 - 2bdef - 2cdef + d^2ef + de^2f + def^2$.

By the assumptions in this case, we have $a - f \geq 0$, $e - b \geq 0$ and $d - c \geq 0$. Therefore, to prove $R \geq 0$, it will be enough to show $T_4 \geq 0$. Again by the assumptions, we can write $a = f + k$, $e = b + m$ and $d = c + n$ for some nonnegative real numbers k, m and n. Now substituting these into T_4 gives $T_4 = b^2kn + 2bck^2 + 2bfn^2 + bk^2n + bkn^2 + c^2km + 2cfm^2 + ck^2m + ckm^2 + f^2mn + fm^2n + fmn^2$, which clearly shows that $T_4 \geq 0$. Hence, $R \geq 0$ in this case.

Case V: Suppose $f \geq a, b \geq e$ and $c \geq d$:

We have

$$\begin{aligned}R = {} & 2\big[af(-b+c+d-e)^2 + be(-a+c+d-f)^2 + cd(-a+b+e-f)^2\big] \\ & +13\big[be(-a-c+d+f)^2 + cd(-a-b+e+f)^2 + af(-b+c-d+e)^2\big] \\ & +15\big[ae(b-f)^2 + ab(e-f)^2 + bf(a-e)^2 + ef(a-b)^2 + df(c-a)^2 \\ & \quad +ad(c-f)^2 + ac(d-f)^2 + cf(a-d)^2 + bd(c-e)^2 \\ & \quad +bc(d-e)^2 + ce(b-d)^2 + de(b-c)^2\big] \\ & +11\big[af(b-e)(c-d) + cd(f-a)(b-e) + be(f-a)(c-d)\big] + 15T_5,\end{aligned}$$

where $T_5 = a^2bd + a^2ce + ab^2d - 2abcd - 2abce + abd^2 - 2abdf + ac^2e + ace^2 - 2acef + b^2cf + bc^2f + bcf^2 - 2bdef - 2cdef + d^2ef + de^2f + def^2$.

By the assumptions in this case, we have $f - a \geq 0$, $b - e \geq 0$ and $c - d \geq 0$. Therefore, to prove $R \geq 0$, it will be enough to show $T_5 \geq 0$. Again by the assumptions, we can write $f = a + k$, $b = e + m$ and $c = d + n$ for some nonnegative real numbers k, m and n. Now substituting these into T_5 gives $T_5 = a^2mn + 2adm^2 + 2aen^2 + 2akmn + am^2n + amn^2 + d^2km + 2dek^2 + dk^2m + dkm^2 + 2dkmn + e^2kn + ek^2n + 2ekmn + ekn^2 + k^2mn + km^2n + kmn^2$. Thus, $T_5 \geq 0$. Hence, $R \geq 0$ in this case.

Case VI: Suppose $f \geq a, b \geq e$ and $d \geq c$:

We have

$$\begin{aligned}R = {} & 2\big[af(-b+c+d-e)^2 + be(-a+c+d-f)^2 + cd(-a+b+e-f)^2\big] \\ & +13\big[be(-a+c-d+f)^2 + cd(-a-b+e+f)^2 + af(-b-c+d+e)^2\big] \\ & +15\big[ae(b-f)^2 + ab(e-f)^2 + bf(a-e)^2 + ef(a-b)^2 + df(c-a)^2 \\ & \quad +ad(c-f)^2 + ac(d-f)^2 + cf(a-d)^2 + bd(c-e)^2 \\ & \quad +bc(d-e)^2 + ce(b-d)^2 + de(b-c)^2\big] \\ & +11\big[af(b-e)(d-c) + cd(f-a)(b-e) + be(f-a)(d-c)\big] + 15T_6,\end{aligned}$$

where $T_6 = a^2bd + a^2ce + ab^2d - 2abcd - 2abcf + abd^2 - 2abde + ac^2e + ace^2 - 2adef + b^2cf + bc^2f - 2bcef + bcf^2 - 2cdef + d^2ef + de^2f + def^2$.

By the assumptions in this case, we have $f - a \geq 0$, $b - e \geq 0$ and $d - c \geq 0$. Therefore, to prove $R \geq 0$, it will be enough to show $T_6 \geq 0$. Again by the assumptions, we can write $f = a+k$, $b = e+m$ and $d = c+n$ for some nonnegative real numbers k, m and n. Now substituting these into T_6 gives $T_6 = a^2mn + 2acm^2 + 2aen^2 + am^2n + amn^2 + c^2km + 2cek^2 + ck^2m + ckm^2 + e^2kn + ek^2n + ekn^2$, so $T_6 \geq 0$. Hence, $R \geq 0$ in this case.

Case VII: Suppose $f \geq a, e \geq b$ and $c \geq d$:

We have

$$\begin{aligned}R = {} & 2\big[af(-b+c+d-e)^2 + be(-a+c+d-f)^2 + cd(-a+b+e-f)^2\big] \\ & +13\big[be(-a-c+d+f)^2 + cd(-a+b-e+f)^2 + af(b+c-d-e)^2\big]\end{aligned}$$

$$
\begin{aligned}
&+15\big[ae(b-f)^2+ab(e-f)^2+bf(a-e)^2+ef(a-b)^2+df(c-a)^2\\
&\qquad +ad(c-f)^2+ac(d-f)^2+cf(a-d)^2+bd(c-e)^2\\
&\qquad +bc(d-e)^2+ce(b-d)^2+de(b-c)^2\big]\\
&+11\big[af(e-b)(c-d)+be(f-a)(c-d)+cd(f-a)(e-b)\big]+15T_7,
\end{aligned}
$$

where $T_7 = a^2bd + a^2ce + ab^2d - 2abce - 2abcf + abd^2 + ac^2e - 2acde + ace^2 - 2adef + b^2cf + bc^2f - 2bcdf + bcf^2 - 2bdef + d^2ef + de^2f + def^2$.

By the assumptions in this case, we have $f-a \geq 0$, $e-b \geq 0$ and $c-d \geq 0$. Therefore, to prove $R \geq 0$, it will be enough to show $T_7 \geq 0$. Again by the assumptions, we can write $f = a+k$, $e = b+m$ and $c = d+n$ for some nonnegative real numbers k, m and n. Now substituting these into T_7 gives $T_7 = a^2mn+2abn^2+2adm^2+am^2n+amn^2+b^2kn+2bdk^2+bk^2n+bkn^2+d^2km+dk^2m+dkm^2$, which clearly shows that $T_7 \geq 0$. Hence, $R \geq 0$ in this case.

Case VIII: Suppose $f \geq a$, $e \geq b$ and $d \geq c$:

We have

$$
\begin{aligned}
R = {}&2\big[af(-b+c+d-e)^2+be(-a+c+d-f)^2+cd(-a+b+e-f)^2\big]\\
&+13\big[be(-a+c-d+f)^2+cd(-a+b-e+f)^2+af(b-c+d-e)^2\big]\\
&+15\big[ae(b-f)^2+ab(e-f)^2+bf(a-e)^2+ef(a-b)^2+df(c-a)^2\\
&\qquad +ad(c-f)^2+ac(d-f)^2+cf(a-d)^2+bd(c-e)^2\\
&\qquad +bc(d-e)^2+ce(b-d)^2+de(b-c)^2\big]\\
&+11\big[af(e-b)(d-c)+be(f-a)(d-c)+cd(f-a)(e-b)\big]+15T_8,
\end{aligned}
$$

where $T_8 = a^2bd + a^2ce + ab^2d + abd^2 - 2abde - 2abdf + ac^2e - 2acde + ace^2 - 2acef + b^2cf + bc^2f - 2bcdf - 2bcef + bcf^2 + d^2ef + de^2f + def^2$.

By the assumptions in this case, we have $f-a \geq 0$, $e-b \geq 0$ and $d-c \geq 0$. Therefore, to prove $R \geq 0$, it will be enough to show $T_8 \geq 0$. Again by the assumptions, we can write $f = a+k$, $e = b+m$ and $d = c+n$ for some nonnegative real numbers k, m and n. Now substituting these into T_8 gives $T_8 = a^2mn + 2abn^2 + 2acm^2 + 2akmn + am^2n + amn^2 + b^2kn + 2bck^2 + bk^2n + 2bkmn + bkn^2 + c^2km + ck^2m + ckm^2 + 2ckmn + k^2mn + km^2n + kmn^2$. This clearly shows that $T_8 \geq 0$. Hence, $R \geq 0$ in this case.

Now, we give more details about reducing these cases by the use of the symmetries of tetrahedral graph. We observe that $\varphi(\Gamma)$ and the expressions A, B and D as in Equation (6) do not change if we make the following substitutions

(i) $a \to f$, $f \to a$, and $b \to e$, $e \to b$,
(ii) $a \to f$, $f \to a$, and $c \to d$, $d \to c$,
(iii) $b \to e$, $e \to b$, and $c \to d$, $d \to c$,

Thus, applying (i) gives that Case I is equivalent to Case VII, and that Case II is equivalent to Case VIII. Similarly, applying (ii) implies that Case I is equivalent to Case VI, and Case II is equivalent to Case V. Finally, applying (iii) shows that Case I is equivalent to Case IV, and Case II is equivalent to Case III. Therefore, it will be enough to consider only the cases I and II.

We note that the number 288 also appears in the volume of a tetrahedron expressed via Cayley-Menger determinant [3, page 98] which is equivalent to the Tartaglia's formula given as a generalization of Heron's formula for the area of a triangle.

Next, we give a summary of the inequalities that we established so far. If $(\Gamma, \mathbf{q})$ is a pm-graph of $\bar{g} = 3$ that is not a single point, then we showed that we have the following equalities and sharp lower bounds:

We have $\lambda(\Gamma) = \frac{2}{7}\ell(\Gamma)$ and $\epsilon(\Gamma) = \frac{5}{3}\ell(\Gamma)$ if $g = 0$, and $\lambda(\Gamma) \geq \frac{3}{28}\ell(\Gamma)$ and $\epsilon(\Gamma) \geq \frac{2}{9}\ell(\Gamma)$ if $1 \leq g \leq 3$.

If $g = 0$, $\varphi(\Gamma) = \frac{4}{3}\ell(\Gamma)$, and if $g = 1$, $\varphi(\Gamma) \geq \frac{1}{9}\ell(\Gamma)$. When $g = 2$, $\varphi(\Gamma) \geq \frac{7}{81}\ell(\Gamma)$. Finally, $\varphi(\Gamma) \geq \frac{17}{288}\ell(\Gamma)$ if $g = 3$.

For any nonnegative six real numbers a, b, c, d, e and f, we showed (for pm-graph of type XIV in Fig. 5) that

$$\frac{15}{32}(a+b+c+d+e+f) - \frac{7(abef+acdf+bcde)+2A}{C} \geq 0, \tag{7}$$

where $A = cd(b+e)(a+f) + be(c+d)(a+f) + af(c+d)(b+e)$ and $C = cd(a+b+e+f)+af(b+c+d+e)+be(a+c+d+f)+abd+ace+bcf+def$.

The equality in (7) holds if $a = b = c = d = e = f > 0$. Similarly, we can rewrite (5) as follows

$$\frac{1}{32}(a+b+c+d+e+f) - \frac{2(abef+acdf+bcde)+A}{C} \geq 0, \tag{8}$$

Moreover, if $a+b+c+d+e+f = 1$, we can rewrite inequalities (7) and (8) as follows:

$$\begin{aligned} &15C - 64A - 224(abef+acdf+bcde) \geq 0, \\ &3C - 16A - 32(abef+acdf+bcde) \geq 0, \end{aligned} \tag{9}$$

where the equalities holds iff $a = b = c = d = e = f = \frac{1}{6}$.

Note that the terms appearing in the equalities above can be interpreted via certain cycles of the tetrahedral graph Γ given in part XIV of Fig. 5. More precisely, those terms can be obtained as below, where we identified an edge by its length:

The non-adjacent (not sharing any vertex) edges of Γ are exactly the pairs a and f, b and e, c and d. The terms in A are formed from these. Namely, the factors of a monomial in A are the product of one of these non-adjacent pairs and the pairwise sum of the other two pairs.

Given a pair of nonadjacent edges, we consider the cycle of Γ that does not pass through the edges of this pair. Namely, the cycles (a, b, f, e), (a, c, f, d) and (b, c, e, d) do not contain the pair of edges (c, d), (b, e) and (a, f), respectively. Multiplying the edges of these cycles gives the monomials in $abfe+acfd+bced$. If the summation of the edges of these cycles are multiplied by the edges of the corresponding pair of edges, we obtain most of the monomials appear in C. The remaining monomials in C are nothing but the product of edges in the cycles of Γ that are given by its 4 faces. Namely, the cycles (a, b, d), (a, c, e), (b, c, f) and (d, e, f).

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