Research Article

Burcu Nişancı Türkmen* and Yılmaz Mehmet Demirci

On a class of Harada rings

https://doi.org/10.1515/math-2022-0548
received July 21, 2022; accepted December 8, 2022

Abstract: In this study, inspired by the definition and a previous study [F. Eryılmaz, *SS-lifting modules and rings*, Miskolc Math. Notes **22** (2021), no. 2, 655–662], left Harada rings are adapted to ss-Harada rings, and the important properties of these rings are provided. The characterization of a left ss-Harada ring R with R left perfect and $\mathrm{Rad}(R)$ included in $\mathrm{Soc}(_R R)$ was found with the help of strongly local R-modules.

Keywords: ss-Harada ring, left Artinian ring, ss-lifting module

MSC 2020: 16D90, 17A01

1 Introduction

Throughout the text, rings are assumed to be associative with identity, and modules are unitary left modules. The terminology and notation we use depend on those of the main references in the theory of rings and modules, such as [1–3]. Other important references are [4,5]. We highlight here some of the specific facts, notations, and terminologies that are used in the text. Let R be a ring and N be an R-module. Notation ($L \subset N$) $L \subseteq N$ implies that L is a (proper) submodule of N. The submodule $L \subseteq N$ is *small* in N, which is indicated by $L \ll N$, if $N \neq L + T$ for each proper submodule T of N ([3, §19.1]). Let L and T be submodules of N. L is *a supplement* of T in N if it is minimal among all submodules of N, satisfying the condition $N = T + L$, or equivalently, $N = T + L$ and $T \cap L \ll L$. If each submodule of N has a supplement in N, the module N is called *supplemented*. A submodule $L \subseteq N$ has ample supplements in N if each submodule T of N where $N = L + T$ includes a supplement of L in N. If every submodule of N has ample supplements in N ([3, §41]), then N is named *amply supplemented*. Characterization of (amply) supplemented modules can be found in [3,6]. If each proper submodule T of a non-zero module N satisfies the condition $T \ll N$, then N is called *hollow*. If the sum of all proper submodules of N forms a proper submodule of N, N is named *local*. A ring R is named *local* if $_R R$ is a local module. Using these definitions, one can easily conclude that local modules are hollow and hollow modules are amply supplemented.

Dual to a small submodule, a submodule L of N is defined to be an *essential submodule* of N in case it satisfies $L \cap T \neq 0$, for all non-zero submodules T of N and is denoted by $L \trianglelefteq N$. If each submodule of N is essential in a direct summand of N, then the module N is called *extending* (see [1]). Dual to the notion of extending modules, a module N is called *lifting* if every submodule L of N lies over a direct summand; in other words, for every $L \subseteq N$, N has a decomposition $N = V \oplus T$ with $V \subseteq L$ and $L \cap T \ll T$. The Socle of a module N is the sum of all simple submodules of N, and the Jacobson radical of N is the sum of all small submodules of N, which are indicated by $\mathrm{Soc}(N)$ and $\mathrm{Rad}(N)$, respectively. Moreover, $\mathrm{Soc}(N)$ is the intersection of every $L \subseteq N$ with $L \trianglelefteq N$.

*** Corresponding author: Burcu Nişancı Türkmen,** Department of Mathematics, Amasya University, Faculty of Art and Science, Ipekköy, Amasya, Turkey, e-mail: burcu.turkmen@amasya.edu.tr
Yılmaz Mehmet Demirci: Department of Engineering Science, Abdullah Gül University, Faculty of Engineering, Kayseri, Turkey, e-mail: yilmaz.demirci@agu.edu.tr

Zhou and Zhang have generalized the notion of Soc(N) to $\mathrm{Soc}_s(N)$, resulting in the class of all simple submodules of N that are small in N instead of the class of all simple submodules of N. Therefore, it directly follows from the definition that $\mathrm{Soc}_s(N)$ is a submodule of both Rad(N) and Soc(N). In [4], authors called a module N *strongly local* if it is local and Rad(N) is a submodule of Soc(N). A module N is *ss-supplemented* if each submodule K of N has an ss-supplement L in N, that is, L is a supplement of K in N and $K \cap L$ is semisimple. M is called *amply ss-supplemented* if each submodule of N has ample ss-supplements in N. Here, a submodule K of N is said to have ample ss-supplements in N in case L includes all ss-supplements of K in N for each submodule L of N satisfying the equality $N = K + L$ (see [4]). The characterization of (amply) ss-supplemented modules via semiperfect rings is also provided in the same work. In [7], semisimple lifting modules are defined as a stronger notion of lifting modules. The module N is *semisimple lifting* if, for each submodule L of N, there exists a decomposition $N = V \oplus T$ such that $V \subseteq L$ and $L \cap T \subseteq \mathrm{Soc}_s(T)$.

Colby and Rutter [8] described *semiprimary QF-3 rings* as rings over which the injective hull of every projective module is projective. In [9], an R-module N is defined to be a *small module* if it is small in its injective hull $E(N)$ and a *non-small module* if it is not a small module. It is clear that any module containing a direct summand other than zero is non-small. In [10], Harada studied the rings that provide the dual property. He characterized them in terms of ideals in [9–11]. Oshiro expressed a *left H-ring* (in honor of Harada) as a left Artinian ring over which each non-small module contains a non-zero injective module and studied the module theoretical properties of left Harada rings. The ring R is a left Harada ring if and only if R is left perfect, and any small cover of an injective module is injective if and only if each injective R-module is lifting. In [12, Theorem 2.11], left Harada rings are characterized as rings R for which one of the following equivalent expressions holds.

(1) Each injective R-module is lifting.
(2) R is left Artinian and any non-small module includes a non-zero injective submodule.
(3) Each R-module is a direct sum of an injective module and a small module.
(4) R is left perfect, and the family of all injective modules is closed under small covers. When this is so, R is then a semiprimary QF-3 ring.

These equivalent conditions are the well-known results that will be considered in the text. Jayaraman and Vanaja introduced the concept of Harada modules (see [13]). They defined a module N to be a Harada module if each injective module in $\sigma[N]$ is lifting. Thus, they extended Oshiro's result above to Harada modules. In [13], they describe the characterizations and some properties of Harada modules.

We study a special concept of left Harada rings, which we call ss-Harada rings. We prove that a ring R is a left ss-Harada ring if and only if every R-module can be expressed as the direct sum of an injective module and a small semisimple module. If R is a left ss-Harada ring, then each injective R-module can be expressed as a direct sum of strongly local R-modules. We provide further equivalent conditions for a ring to be a left ss-Harada ring in case the ring at hand is left perfect with Rad(R) contained in $\mathrm{Soc}({}_R R)$.

2 ss-Harada rings

In this section, we start by defining ss-Harada rings that will help us alter the result of Oshiro given in introduction.

Definition 2.1. A ring R is called *left ss-Harada* if each injective left R-module is semisimple lifting.

Clearly, every left ss-Harada ring is left Harada, but it is not generally true that every left Harada ring is a left ss-Harada ring. Later, we shall provide an example of such rings (see Example 2.7).

Let us consider the following statements for a ring R with corresponding abbreviations that will be used throughout the text.

(*) Every non-small R-module includes a non-zero injective submodule.

(ICC) The family of injective R-modules is closed under small covers, i.e., for every injective R-module E, exact sequence $K \xrightarrow{\psi} E \longrightarrow 0$ of R-modules, and R-module homomorphisms with $\mathrm{Ker}(\psi) \ll K$, we have K is injective.

(ISD) Every R-module can be expressed as a direct sum of an injective module and a small module.

It is obvious that the condition $(*)$ is equivalent to the following: Let N be an injective R-module and K be a submodule of N such that it is non-small in N. So K contains a non-zero direct summand of N. Using this fact, we obtain that each left Harada ring provides the condition $(*)$.

Now, we provide a characterization of left ss-Harada rings. Note that without the restriction of generality we will assume that $K \subseteq E(K)$ for every module K over any ring R.

Lemma 2.2. *Let R be a left ss-Harada ring and M be a non-small R module. Then, there exists a decomposition $M = X \oplus Y$ such that X is a non-zero injective submodule and $Y \subseteq \mathrm{Soc}_s(E(M))$.*

Proof. Since R is a left ss-Harada ring, there exist submodules $X \subseteq M$ and Z of $E(M)$ such that $E(M) = X \oplus Z$ and $M \cap Z \subseteq \mathrm{Soc}_s(E(M))$. Put $Y = M \cap Z$. So X is injective and $Y \subseteq \mathrm{Soc}_s(E(M))$. Using the modular law, we have the decomposition $M = X \oplus Y$ as required. □

Lemma 2.3. *If R is a left ss-Harada ring, then R is a left Artinian ring.*

Proof. It follows from the fact that left Harada rings are left Artinian. □

Theorem 2.4. *Let R be a ring. Then, R is a left ss-Harada ring if and only if every R-module can be expressed as a direct sum of an injective module and a small semisimple module.*

Proof. (⇒) Let M be an R-module. Since R is a left ss-Harada ring, there exists a decomposition $E(M) = X \oplus Y$ such that $X \subseteq M$ and $Y \cap M \subseteq \mathrm{Soc}_s(E(M))$. By the proof of Lemma 2.2, it follows.

(⇐) Let E be an injective R-module and M be any submodule of E. By assumption, there exists the decomposition $M = X \oplus Y$, where X is injective and Y is a small semisimple module. Clearly, X is also a direct summand of E and Y is a small submodule of $E(Y)$. Therefore, $Y \subseteq \mathrm{Soc}_s(E)$ since E is injective. It follows from [7, Theorem 1] that E is semisimple lifting. Hence, R is a left ss-Harada ring. □

Corollary 2.5. *Let R be a left ss-Harada ring and M be an R-module. Then, M is ss-supplemented and* $\mathrm{Rad}(M)$ *is semisimple.*

Proof. By Theorem 2.4, we have $M = X \oplus Y$, where X is an injective module and Y is a semisimple module. Since injective modules are ss-supplemented, M is ss-supplemented as a direct sum of two ss-supplemented modules according to [4, Corollary 24]. It follows from [4, Theorem 41] that the radical $\mathrm{Rad}(M)$ is semisimple. □

The following fact is a consequence of Corollary 2.5 and Lemma 2.3.

Corollary 2.6. *If R is a left ss-Harada ring, R is a left Artinian ring with semisimple radical.*

Recall from [1, 2.17] that a module N is called *uniserial* if the lattice of submodules of N is linearly ordered by inclusion. If ${}_RR$ (R_R) is a uniserial R-module, the ring R is called *left* (*right*) *serial*. Using Corollary 2.6, we provide an example of ring, which is left Harada, but not left ss-Harada.

Example 2.7. Let $R = \mathbb{Z}_8$. Then, $\mathrm{Rad}(R) = \{\overline{0}, \overline{2}, \overline{4}, \overline{6}\}$ and $\mathrm{Soc}(R) = \{\overline{0}, \overline{4}\}$. It is well known that R is a serial Artinian ring. Therefore, the ring is Harada. Since $\mathrm{Rad}(R)$ is not semisimple, R is not ss-Harada by Corollary 2.6.

Proposition 2.8. *If R is a left ss-Harada ring, then each injective R-module can be expressed as a direct sum of strongly local R-modules.*

Proof. Let R be a left ss-Harada ring and E be any injective R-module. It follows from [12, Proposition 2.6] that E is the direct sum of local R-modules, that is, $E = \oplus_{i \in I} L_i$, where each L_i is a local R-module and I is some index set. Therefore, L_i is a semisimple lifting module as a direct summand of E for every $i \in I$. By [4, Proposition 16], L_i $(i \in I)$ is a strongly local module. This completes the proof. □

Let M be a module. M is called *coatomic* if every proper submodule of M is contained in a maximal submodule of M.

Corollary 2.9. *Let R be a left ss-Harada ring and E be an injective R-module. Then, E is coatomic.*

Proof. By [4, Theorem 27] and Proposition 2.8.

We show that the class of left ss-Harada rings is non-empty in the first part of the following example. We also show that there are rings that are not left ss-Harada. □

Example 2.10.
(1) Let R be the remainder class ring of $\mathbb{Z}$ with respect to modulo 4. The sets $\mathrm{Rad}(R)$ and $\mathrm{Soc}({}_R R)$ are equal to the set $\{\overline{0}, \overline{2}\}$. Since R is a left and right serial ring, every injective R-module is semisimple lifting by [7, Theorem 6]. Therefore, $R = \mathbb{Z}_4$ is a left ss-Harada ring.
(2) Suppose that R is a local Dedekind domain and K is the quotient field of R. Then, ${}_R K$ is injective but not semisimple lifting by [7, Example 1]. So, R is not a left ss-Harada ring.

Finally, the characterization of left ss-Harada rings in terms of injective and small semisimple modules is given in the following theorem.

Theorem 2.11. *For a left perfect ring R with* $\mathrm{Rad}(R)$ *a submodule of* $\mathrm{Soc}({}_R R)$, *the following statements are equivalent*:
(1) *R is a left ss-Harada ring.*
(2) *Each cyclic R-module is a direct sum of an injective module and a small semisimple module.*
(3) *Each strongly local module is either injective or small semisimple.*

Proof. (1) ⇒ (3) It follows from [12, Remark of Theorem 2.11] and [7, Lemma 4].

(2) ⇒ (3) Clear by [14, Proposition 2].

(3) ⇒ (1) R satisfies the condition $(*)$ by using [14, Proposition 2]. If we consider [9, Theorem 5], we can say R is a left Harada ring. Moreover, R is a left ss-Harada ring by [7, Lemma 4]. □

Theorem 2.12. *Let R be a ring. Then, the following statements are equivalent*:
(1) *Every injective left R-module is a direct sum of semisimple lifting modules.*
(2) *R is a left Noetherian ring and its indecomposable injective modules are strongly local.*

Proof. (1) ⇒ (2) By [15, Theorem 1], R is a left Noetherian ring. Let M be an indecomposable injective R-module. It follows from (1) that M is semisimple lifting. Therefore, it is strongly local.

(2) ⇒ (1) Let M be an injective R-module. Since R is a left Noetherian ring, M is a direct sum of indecomposable R-modules. By (2), the proof is completed. □

Acknowledgement: The authors thank the referees for their useful comments.

Conflict of interest: The authors state that there is no conflict of interest.

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