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POSITIVE SOLUTIONS OF THE p-LAPLACIAN DYNAMIC EQUATIONS ON TIME SCALES WITH SIGN CHANGING NONLINEARITY

ABDULKADIR DOGAN

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Abstract. This article concerns the existence of positive solutions for p-Laplacian boundary value problem on time scales. By applying fixed point index we obtain the existence of solutions. Emphasis is put on the fact that the nonlinear term is allowed to change sign. An example illustrates our results.

1. Introduction

The development of the theory of time scales was initiated by Stefan Hilger in his Ph.D. thesis in 1988 [12] as a means of a unifying structure for the study of differential equations in the continuous case and the study of finite difference equations in the discrete case. The study of time scales has led to several important applications, e.g., in the study of insect population models, heat transfer, neural networks, phytoremediation of metals, wound healing, and epidemic models [5, 14, 19, 24].

In the last few years, there is much attention focused on the existence of positive solutions for second-order boundary value problems (BVPs) on time scales [2, 6, 7, 8, 11, 16, 17, 18, 21, 22, 23, 25, 27]. But for the existence of positive solutions for p-Laplacian BVPs with sign changing nonlinearity on time scales [20, 28], few works were done as far as we know.

Agarwal, Lü and O'Regan [1] studied the singular BVP

$$(\varphi_p(y'))' + q(t)f(t,y) = 0, \quad 0 < t < 1,$$
$$y(0) = y(1) = 0,$$

by means of the upper and lower solution method, where the nonlinearity f is allowed to change sign.

Anderson, Avery and Henderson[3] studied the BVP

$$(g(u^\Delta))^\nabla + c(t)f(u) = 0, \quad a < t < b,$$
$$u(a) - B_0(u^\Delta(\nu)) = 0, \quad u^\Delta(b) = 0,$$

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where $g(z) = |z|^{p-2}z,\ p > 1,\ \nu \in (a,b),\ f \in C_{ld}([0,+\infty),[0,+\infty)),\ c(t) \in C_{ld}([a,b],[0,+\infty))$ and $K_m x \le B_0(x) \le K_M x$ for some positive constants K_m, K_M. They proved the existence of at least one positive solution by using a fixed point theorem of cone expansion and compression of functional type.

Ji, Feng and Ge [13] considered the existence of multiple positive solutions to BVP for the one-dimensional p-Laplacian equation

$$\begin{gathered}(\phi_p(u'(t)))' + q(t)f(t,u(t)) = 0, \quad t \in (0,1),\\ u(0) = \sum_{i=1}^{n} \alpha_i u(\xi_i), \quad u(1) = \sum_{i=1}^{n} \beta_i u(\xi_i),\end{gathered}$$

where $0 < \xi_1 < \xi_2 < \dots < \xi_n < 1, \alpha_i, \beta_i \in [0,\infty)$ satisfy $0 < \sum_{i=1}^{n} \alpha_i, \sum_{i=1}^{n} \beta_i < 1$. The nonlinear term f may change sign. By applying a fixed-point theorem for operators in a cone, they provided sufficient conditions for the existence of multiple positive solutions to the BVP.

In [18], Sang, Su and Xu studied the existence of positive solutions of the following dynamic equations on time scales:

$$\begin{gathered}(\phi(u^{\Delta}))^{\nabla} + a(t)f(t,u(t)) = 0, \quad t \in [0,T],\\ \phi(u^{\Delta}(0)) = \sum_{i=1}^{m-2} a_i \phi(u^{\Delta}(\xi_i)), \quad u(T) = \sum_{i=1}^{m-2} b_i u(\xi_i),\end{gathered}$$

where $\phi : R \to R$ is an increasing homeomorphism and homomorphism and $\phi(0) = 0$. They established several existence theorems of positive solutions for nonlinear m-point BVP. The nonlinearity f is allowed to change sign.

Su, Li and Sun [20] investigated the following singular m-point p-Laplacian BVP with the sign changing nonlinearity on time scales:

$$\begin{gathered}(\varphi_p(u^{\Delta}(t)))^{\nabla} + q(t)f(t,u(t)) = 0, \quad t \in (0,T)_{\mathbb{T}},\\ u(0) = 0, \quad u(T) - \sum_{i=1}^{m-2} \psi_i(u(\xi_i)) = 0,\end{gathered}$$

where $\varphi_p(u) = |u|^{p-2}u,\ p > 1,\ 0 < \xi_1 < \xi_2 < \dots < \xi_{m-2} < \rho(T),\ q(t) \in C_{ld}((0,T),(0,+\infty)),\ f \in C_{ld}((0,T) \times (0,+\infty),(-\infty,+\infty))$. They presented some new existence criteria for positive solutions of the problem by using the well-known Schauder fixed point theorem and upper and lower solutions method.

Wang and Hou [26] considered the multipoint BVP

$$\begin{gathered}(\phi_p(u'(t)))' + f(t,u(t)) = 0, \quad t \in (0,1),\\ \phi_p(u'(0)) = \sum_{i=1}^{n-2} a_i \phi_p(u'(\xi_i)), \quad u(1) = \sum_{i=1}^{n-2} \beta_i u(\xi_i),\end{gathered}$$

where $\phi_p(u) = |u|^{p-2}u$, $p > 1$, $\xi_i \in (0,1)$ with $0 < \xi_1 < \xi_2 < \dots < \xi_{n-2} < 1$, and a_i, b_i satisfy $a_i, b_i \in [0,\infty)$, $0 < \sum_{i=1}^{n-2} a_i < 1$, and $0 < \sum_{i=1}^{n-2} b_i < 1$. Using a fixed point theorem for operators on a cone, they provided sufficient conditions for the existence of at least three positive solutions to the above BVP.

Motivated by works mentioned above, in this paper, we are concerned with the existence of multiple positive solutions for p-Laplacian multipoint BVP on time scales

$$(\phi_p(u^\Delta(t)))^\nabla + w(t)f(t,u(t)) = 0, \quad t \in [0,T]_{\mathbb{T}}, \tag{1.1}$$

$$u(0) = \sum_{i=1}^{m-2} a_i u(\xi_i), \quad \phi_p(u^\Delta(T)) = \sum_{i=1}^{m-2} b_i \phi_p(u^\Delta(\xi_i)), \tag{1.2}$$

where $\phi_p(u)$ is p-Laplacian operator, i.e., $\phi_p(u) = |u|^{p-2}u$, for $p > 1$ with $(\phi_p)^{-1} = \phi_q$ and $\frac{1}{p} + \frac{1}{q} = 1$, $0 < \xi_1 < \dots < \xi_{m-2} < \rho(T)$. The usual notation and terminology for time scales as can be found in [4, 5, 9], will be used here.

We need the following hypotheses:

- (H1) $a_i, b_i \in [0,+\infty)$ satisfy $\sum_{i=1}^{m-2} a_i < 1$ and $0 < \sum_{i=1}^{m-2} b_i < 1$;
- (H2) $w \in C_{ld}([0,T]_{\mathbb{T}}, [0,+\infty))$ with $\int_0^{\xi_1} w(\tau)\nabla\tau > 0$;
- (H3) $f : [0,T]_{\mathbb{T}} \times [0,+\infty) \to R$ is continuous.

We find some new results on the existence of at least two positive solutions to the BVP (1.1) and (1.2) by using fixed point index. The interesting point of this article is that the nonlinear term f is allowed to change sign.

2. Preliminaries

Let the Banach space $E = C_{ld}([0,T]_{\mathbb{T}}, \mathbb{R})$ with norm $\|u\| = \sup_{t\in[0,T]_{\mathbb{T}}} |u(t)|$, and define two cones:

$$P = \{u : u \in E, u(t) \geq 0, t \in [0,T]_{\mathbb{T}}\},$$

$$P' = \{u : u \in E, u \text{ is concave, nonnegative and increasing on } [0,T]_{\mathbb{T}}\}.$$

Lemma 2.1. *If $1-\sum_{i=1}^{m-2} a_i \neq 0$ and $1-\sum_{i=1}^{m-2} b_i \neq 0$, then for $h \in C_{ld}([0,T]_{\mathbb{T}}, R)$, the BVP*

$$(\phi_p(u^\Delta(t)))^\nabla + h(t) = 0, \quad t \in (0,T)_{\mathbb{T}}, \tag{2.1}$$

$$u(0) = \sum_{i=1}^{m-2} a_i u(\xi_i), \quad \phi_p(u^\Delta(T)) = \sum_{i=1}^{m-2} b_i \phi_p(u^\Delta(\xi_i)) \tag{2.2}$$

has a unique solution for which the following representation holds

$$\begin{aligned} u(t) &= \int_t^T \phi_q\Big(\int_0^s h(\tau)\nabla\tau - C\Big)\Delta s \\ &\quad + \frac{1}{1-\sum_{i=1}^{m-2} a_i}\Big[\sum_{i=1}^{m-2} a_i \int_{\xi_i}^T \phi_q\Big(\int_0^s h(\tau)\nabla\tau - C\Big)\Delta s \\ &\quad - \int_0^T \phi_q\Big(\int_0^s h(\tau)\nabla\tau - C\Big)\Delta s\Big], \end{aligned} \tag{2.3}$$

where

$$C = \frac{-1}{1-\sum_{i=1}^{m-2} b_i}\Big[\sum_{i=1}^{m-2} b_i \int_0^{\xi_i} h(\tau)\nabla\tau - \int_0^T h(\tau)\nabla\tau\Big].$$

Proof. Integrating (2.1) from 0 to t, one gets

$$\phi_p(u^\Delta(t)) - \phi_p(u^\Delta(0)) = -\int_0^t h(\tau)\nabla\tau.$$

Let $C=\phi_p(u^\Delta(0))$, then we can write

$$\phi_p(u^\Delta(t))=-\int_0^t h(\tau)\nabla\tau+C, \tag{2.4}$$

i.e.

$$u^\Delta(t)=\phi_q\Big(-\int_0^t h(\tau)\nabla\tau+C\Big). \tag{2.5}$$

Integrating the dynamic equation (2.5) from t to T, we have

$$u(T)-u(t)=\int_t^T \phi_q\Big(-\int_0^s h(\tau)\nabla\tau+C\Big)\Delta s.$$

Then we obtain

$$u(t)=u(T)+\int_t^T \phi_q\Big(\int_0^s h(\tau)\nabla\tau-C\Big)\Delta s. \tag{2.6}$$

Applying the first boundary condition, one gets

$$\begin{aligned} u(0)&=u(T)+\int_0^T \phi_q\Big(\int_0^s h(\tau)\nabla\tau-C\Big)\Delta s\\ &=\sum_{i=1}^{m-2}a_iu(\xi_i)=\sum_{i=1}^{m-2}a_i\Big[u(T)+\int_{\xi_i}^T \phi_q\Big(\int_0^s h(\tau)\nabla\tau-C\Big)\Delta s\Big], \end{aligned}$$

i.e.,

$$\begin{aligned} u(T)=\frac{1}{1-\sum_{i=1}^{m-2}a_i}\Big[&\sum_{i=1}^{m-2}a_i\int_{\xi_i}^T \phi_q\Big(\int_0^s h(\tau)\nabla\tau-C\Big)\Delta s\\ &-\int_0^T \phi_q\Big(\int_0^s h(\tau)\nabla\tau-C\Big)\Delta s\Big]. \end{aligned} \tag{2.7}$$

Therefore, by (2.6) and (2.7), we have

$$\begin{aligned} u(t)=\frac{1}{1-\sum_{i=1}^{m-2}a_i}\Big[&\sum_{i=1}^{m-2}a_i\int_{\xi_i}^T \phi_q\Big(\int_0^s h(\tau)\nabla\tau-C\Big)\Delta s\\ &-\int_0^T \phi_q\Big(\int_0^s h(\tau)\nabla\tau-C\Big)\Delta s\Big]+\int_t^T \phi_q\Big(\int_0^s h(\tau)\nabla\tau-C\Big)\Delta s. \end{aligned}$$

Applying the second boundary condition in (2.4), we have

$$\begin{aligned} \phi_p(u^\Delta(T))&=-\int_0^T h(\tau)\nabla\tau+C\\ &=\sum_{i=1}^{m-2}b_i\phi_p(u^\Delta(\xi_i))=\sum_{i=1}^{m-2}b_i\Big(-\int_0^{\xi_i} h(\tau)\nabla\tau+C\Big). \end{aligned} \tag{2.8}$$

Solving the above equation, we obtain

$$C=\frac{-1}{1-\sum_{i=1}^{m-2}b_i}\Big[\sum_{i=1}^{m-2}b_i\int_0^{\xi_i} h(\tau)\nabla\tau-\int_0^T h(\tau)\nabla\tau\Big].$$

Let u_1 and u_2 be two solutions of problem (2.1), (2.2). Then, using representation (2.3), we obtain $u_1(t)-u_2(t)=0$, $t\in[0,T]$. Thus u in (2.3) is the unique solution of (2.1) and (2.2). The proof is complete. □

Lemma 2.2. *Suppose* (H1) *holds, for* $h \in C_{ld}([0,T]_{\mathbb{T}}, R)$ *and* $h(t) \geq 0, t \in [0,T]$. *Then, the unique solution* u *of* (2.1) *and* (2.2) *satisfies* $u(t) \geq 0$, *for* $t \in [0,T]_{\mathbb{T}}$.

Proof. Let

$$\varphi_0(s) = \phi_q\Big(\int_0^s h(\tau)\nabla\tau - C\Big).$$

Since

$$\begin{aligned}
&\int_0^s h(\tau)\nabla\tau + \frac{\sum_{i=1}^{m-2} b_i \int_0^{\xi_i} h(\tau)\nabla\tau - \int_0^T h(\tau)\nabla\tau}{1-\sum_{i=1}^{m-2} b_i}\\
&= \int_0^s h(\tau)\nabla\tau + \frac{\sum_{i=1}^{m-2} b_i\Big(\int_0^T h(\tau)\nabla\tau + \int_T^{\xi_i} h(\tau)\nabla\tau\Big) - \int_0^T h(\tau)\nabla\tau}{1-\sum_{i=1}^{m-2} b_i}\\
&= \int_0^s h(\tau)\nabla\tau - \frac{\Big(\int_0^T h(\tau)\nabla\tau - \sum_{i=1}^{m-2} b_i \int_0^T h(\tau)\nabla\tau\Big)}{1-\sum_{i=1}^{m-2} b_i} - \frac{\sum_{i=1}^{m-2} b_i \int_{\xi_i}^T h(\tau)\nabla\tau}{1-\sum_{i=1}^{m-2} b_i}\\
&= \int_0^s h(\tau)\nabla\tau - \frac{\Big(1-\sum_{i=1}^{m-2} b_i\Big)\int_0^T h(\tau)\nabla\tau}{1-\sum_{i=1}^{m-2} b_i} - \frac{\sum_{i=1}^{m-2} b_i \int_{\xi_i}^T h(\tau)\nabla\tau}{1-\sum_{i=1}^{m-2} b_i}\\
&= \int_0^s h(\tau)\nabla\tau - \int_0^T h(\tau)\nabla\tau - \frac{\sum_{i=1}^{m-2} b_i \int_{\xi_i}^T h(\tau)\nabla\tau}{1-\sum_{i=1}^{m-2} b_i}\\
&= -\int_s^T h(\tau)\nabla\tau - \frac{\sum_{i=1}^{m-2} b_i \int_{\xi_i}^T h(\tau)\nabla\tau}{1-\sum_{i=1}^{m-2} b_i} \leq 0,
\end{aligned}$$

it follows that $\varphi_0(s) \leq 0$. According to Lemma 2.1, we obtain

$$\begin{aligned}
u(0) &= \int_0^T \varphi_0(s)\Delta s + \frac{1}{1-\sum_{i=1}^{m-2} a_i}\Big[\sum_{i=1}^{m-2} a_i \int_{\xi_i}^T \varphi_0(s)\Delta s - \int_0^T \varphi_0(s)\Delta s\Big]\\
&= \frac{-\sum_{i=1}^{m-2} a_i \int_0^T \varphi_0(s)\Delta s + \sum_{i=1}^{m-2} a_i \int_{\xi_i}^T \varphi_0(s)\Delta s}{1-\sum_{i=1}^{m-2} a_i}\\
&= \frac{-\sum_{i=1}^{m-2} a_i\Big[\int_0^{\xi_i} \varphi_0(s)\Delta s + \int_{\xi_i}^T \varphi_0(s)\Delta s\Big] + \sum_{i=1}^{m-2} a_i \int_{\xi_i}^T \varphi_0(s)\Delta s}{1-\sum_{i=1}^{m-2} a_i}\\
&= \frac{-\sum_{i=1}^{m-2} a_i \int_0^{\xi_i} \varphi_0(s)\Delta s}{1-\sum_{i=1}^{m-2} a_i} \geq 0
\end{aligned}$$

and

$$\begin{aligned}
u(T) &= \frac{1}{1-\sum_{i=1}^{m-2} a_i}\Big[\sum_{i=1}^{m-2} a_i \int_{\xi_i}^T \varphi_0(s)\Delta s - \int_0^T \varphi_0(s)\Delta s\Big]\\
&= \frac{\sum_{i=1}^{m-2} a_i \int_{\xi_i}^T \varphi_0(s)\Delta s - \int_0^{\xi_i} \varphi_0(s)\Delta s - \int_{\xi_i}^T \varphi_0(s)\Delta s}{1-\sum_{i=1}^{m-2} a_i}\\
&= \frac{-\Big(1-\sum_{i=1}^{m-2} a_i\Big)\int_{\xi_i}^T \varphi_0(s)\Delta s - \int_0^{\xi_i} \varphi_0(s)\Delta s}{1-\sum_{i=1}^{m-2} a_i}
\end{aligned}$$

$$= -\int_{\xi_i}^{T} \varphi_0(s)\Delta s - \frac{\int_0^{\xi_i} \varphi_0(s)\Delta s}{1-\sum_{i=1}^{m-2} a_i} \geq 0.$$

If $t \in (0,T)_{\mathbb{T}}$, we have

$$\begin{aligned}
&u(t)\\
&= \int_t^T \varphi_0(s)\Delta s + \frac{1}{1-\sum_{i=1}^{m-2} a_i}\Big[\sum_{i=1}^{m-2} a_i \int_{\xi_i}^T \varphi_0(s)\Delta s - \int_0^T \varphi_0(s)\Delta s\Big]\\
&= \frac{1}{1-\sum_{i=1}^{m-2} a_i}\Big[\int_t^T \varphi_0(s)\Delta s - \sum_{i=1}^{m-2} a_i \int_t^T \varphi_0(s)\Delta s\\
&\quad + \sum_{i=1}^{m-2} a_i \int_{\xi_i}^T \varphi_0(s)\Delta s - \int_0^T \varphi_0(s)\Delta s\Big]\\
&\geq \frac{1}{1-\sum_{i=1}^{m-2} a_i}\Big[\int_0^T \varphi_0(s)\Delta s - \sum_{i=1}^{m-2} a_i \int_0^T \varphi_0(s)\Delta s\\
&\quad + \sum_{i=1}^{m-2} a_i \int_{\xi_i}^T \varphi_0(s)\Delta s - \int_0^T \varphi_0(s)\Delta s\Big]\\
&= \frac{1}{1-\sum_{i=1}^{m-2} a_i}\Big[-\sum_{i=1}^{m-2} a_i\Big(\int_0^{\xi_i} \varphi_0(s)\Delta s + \int_{\xi_i}^T \varphi_0(s)\Delta s\Big) + \sum_{i=1}^{m-2} a_i \int_{\xi_i}^T \varphi_0(s)\Delta s\Big]\\
&= \frac{-\sum_{i=1}^{m-2} a_i \int_0^{\xi_i} \varphi_0(s)\Delta s}{1-\sum_{i=1}^{m-2} a_i} \geq 0.
\end{aligned}$$

Therefore $u(t) \geq 0$, $t \in [0,T]_{\mathbb{T}}$. □

Lemma 2.3. *If $u \in P'$ and it satisfies* (1.2)*, then*

$$\inf_{t\in[0,T]_{\mathbb{T}}} u(t) \geq \gamma_1 \|u\|,$$

where $\gamma_1 = \sum_{i=1}^{m-2} a_i \frac{\xi_i}{T}$, $\|u\| = \max_{t\in[0,T]_{\mathbb{T}}} |u(t)|$.

Proof. Since $u^{\Delta\nabla}(t) \leq 0$, it follows that $u^{\Delta}(t)$ is nonincreasing. Thus, for $0 < t < T$,

$$u(t) - u(0) = \int_0^t u^{\Delta}(s)\Delta s \geq t u^{\Delta}(t)$$

and

$$u(T) - u(t) = \int_t^T u^{\Delta}(s)\Delta s \leq (T-t)u^{\Delta}(t)$$

from which we have

$$u(t) \geq \frac{tu(T) + (T-t)u(0)}{T} \geq \frac{t}{T}u(T) = \frac{t}{T}\|u\|,$$

and so $u(\xi_i) \geq \frac{\xi_i}{T}\|u\|$.

With the boundary condition $u(0) = \sum_{i=1}^{m-2} a_i u(\xi_i)$, we obtain

$$\inf_{t\in[0,T]_{\mathbb{T}}} u(t) = u(0) = \sum_{i=1}^{m-2} a_i u(\xi_i) \geq \sum_{i=1}^{m-2} a_i \frac{\xi_i}{T}\|u\| = \gamma_1\|u\|.$$

This completes the proof. □

Let

$$K = \Big\{u : u \in E, u \text{ is nonnegative and increasing on } [0,T]_{\mathbb{T}},\ \min_{t\in[0,T]_{\mathbb{T}}} u(t) \geq \gamma\|u\|\Big\},$$

where $\gamma = \gamma_1\gamma_2$,

$$\gamma_2 = \frac{\sum_{i=1}^{m-2} a_i \int_0^{\xi_i} \phi_q\Big(\int_s^T w(\tau)\nabla\tau + \frac{\sum_{i=1}^{m-2} b_i \int_{\xi_i}^T w(\tau)\nabla\tau}{1-\sum_{i=1}^{m-2} b_i}\Big)\Delta s}{\int_0^T \phi_q\Big(\int_s^T w(\tau)\nabla\tau + \frac{\sum_{i=1}^{m-2} b_i \int_{\xi_i}^T w(\tau)\nabla\tau}{1-\sum_{i=1}^{m-2} b_i}\Big)\Delta s}.$$

Note that u is a solution of BVP (1.1) and (1.2) if and only if

$$\begin{aligned} u(t) &= \int_t^T \phi_q\Big(\int_0^s w(\tau)f(\tau,u(\tau))\nabla\tau - C\Big)\Delta s \\ &\quad + \frac{1}{1-\sum_{i=1}^{m-2} a_i}\Big[\sum_{i=1}^{m-2} a_i \int_{\xi_i}^T \phi_q\Big(\int_0^s w(\tau)f(\tau,u(\tau))\nabla\tau - C\Big)\Delta s \\ &\quad - \int_0^T \phi_q\Big(\int_0^s w(\tau)f(\tau,u(\tau))\nabla\tau - C\Big)\Delta s\Big], \end{aligned}$$

where

$$C = \frac{-1}{1-\sum_{i=1}^{m-2} b_i}\Big[\sum_{i=1}^{m-2} b_i \int_0^{\xi_i} w(\tau)f(\tau,u(\tau))\nabla\tau - \int_0^T w(\tau)f(\tau,u(\tau))\nabla\tau\Big].$$

We define the operators $A : P \to E$ and $F : K \to E$ as follows

$$\begin{aligned} (Au)(t) &= \int_t^T \phi_q\Big(\int_0^s w(\tau)f(\tau,u(\tau))\nabla\tau - C\Big)\Delta s \\ &\quad + \frac{1}{1-\sum_{i=1}^{m-2} a_i}\Big[\sum_{i=1}^{m-2} a_i \int_{\xi_i}^T \phi_q\Big(\int_0^s w(\tau)f(\tau,u(\tau))\nabla\tau - C\Big)\Delta s \\ &\quad - \int_0^T \phi_q\Big(\int_0^s w(\tau)f(\tau,u(\tau))\nabla\tau - C\Big)\Delta s\Big], \end{aligned} \tag{2.9}$$

where

$$C = \frac{-1}{1-\sum_{i=1}^{m-2} b_i}\Big[\sum_{i=1}^{m-2} b_i \int_0^{\xi_i} w(\tau)f(\tau,u(\tau))\nabla\tau - \int_0^T w(\tau)f(\tau,u(\tau))\nabla\tau\Big],$$

$$\begin{aligned} (Fu)(t) &= \int_t^T \phi_q\Big(\int_0^s w(\tau)f^+(\tau,u(\tau))\nabla\tau - \tilde{C}\Big)\Delta s \\ &\quad + \frac{1}{1-\sum_{i=1}^{m-2} a_i}\Big[\sum_{i=1}^{m-2} a_i \int_{\xi_i}^T \phi_q\Big(\int_0^s w(\tau)f^+(\tau,u(\tau))\nabla\tau - \tilde{C}\Big)\Delta s \\ &\quad - \int_0^T \phi_q\Big(\int_0^s w(\tau)f^+(\tau,u(\tau))\nabla\tau - \tilde{C}\Big)\Delta s\Big], \end{aligned} \tag{2.10}$$

where

$$\tilde{C} = \frac{-1}{1-\sum_{i=1}^{m-2} b_i}\Big[\sum_{i=1}^{m-2} b_i \int_0^{\xi_i} w(\tau)f^+(\tau,u(\tau))\nabla\tau - \int_0^T w(\tau)f^+(\tau,u(\tau))\nabla\tau\Big],$$

$$f^+(t,u(t)) = \max\{f(t,u(t)),0\}, \quad t \in [0,T]_{\mathbb{T}}.$$

Lemma 2.4. *The operator $F : K \to K$ is completely continuous.*

Proof. Firstly, we show that $F(K) \subset K$. for all $u \in K$, it is easy to see that Fu is nonnegative, concave and increasing on $[0,T]_{\mathbb{T}}$. Thus, $Fu \in K$. Moreover, we know that Fu satisfies (2.2). Hence, Lemma 2.3 implies

$$\inf_{t\in[0,T]_{\mathbb{T}}} (Fu)(t) \geq \gamma_1 \|Fu\|, \quad \text{for } u \in K,$$

i.e., $Fu \in K$. Therefore, we can find that $F(K) \subset K$.

Secondly, we show that F maps bounded set into itself. Suppose that $c > 0$ is a constant and $u \in \overline{K_c} = \{u \in K : \|u\| \leq c\}$. Note that the continuity of f^+ guarantees that there is a $L > 0$ such that $f^+(t,u(t)) \leq \phi_p(L)$ for $t \in [0,T]_{\mathbb{T}}$. Therefore,

$$\begin{aligned}
\|Fu\| &= \max_{t\in[0,T]_{\mathbb{T}}} Fu(t) \\
&\leq \frac{\sum_{i=1}^{m-2} a_i \int_{\xi_i}^T \phi_q\big(\int_0^s w(\tau)f^+(\tau,u(\tau))\nabla\tau - \tilde{C}\big)\Delta s}{1-\sum_{i=1}^{m-2} a_i} \\
&\quad - \frac{\int_0^T \phi_q\big(\int_0^s w(\tau)f^+(\tau,u(\tau))\nabla\tau - \tilde{C}\big)\Delta s}{1-\sum_{i=1}^{m-2} a_i} \\
&\leq -\frac{\int_0^T \phi_q\big(\int_0^s w(\tau)f^+(\tau,u(\tau))\nabla\tau - \tilde{C}\big)\Delta s}{1-\sum_{i=1}^{m-2} a_i} \\
&\leq L\int_0^T \phi_q\Big(\int_s^T w(\tau)\nabla\tau + \frac{\sum_{i=1}^{m-2} b_i \int_{\xi_i}^T w(\tau)\nabla\tau}{1-\sum_{i=1}^{m-2} a_i}\Big)\Delta s,
\end{aligned}$$

where $\phi_q\big(f^+(\tau,u(\tau))\big) \leq L$. That is, $F\overline{K_c}$ is uniformly bounded.

In addition, for any $t_1, t_2 \in [0,T]_{\mathbb{T}}$, we have

$$\begin{aligned}
|Fu(t_1) - Fu(t_2)| &= \Big|\int_{t_1}^T \phi_q\Big(\int_0^s w(\tau)f^+(\tau,u(\tau))\nabla\tau - \tilde{C}\Big)\Delta s \\
&\quad - \int_{t_2}^T \phi_q\Big(\int_0^s w(\tau)f^+(\tau,u(\tau))\nabla\tau - \tilde{C}\Big)\Delta s\Big| \\
&= \Big|\int_{t_1}^{t_2} \phi_q\Big(\int_0^s w(\tau)f^+(\tau,u(\tau))\nabla\tau - \tilde{C}\Big)\Delta s\Big| \\
&\leq L|t_1 - t_2|\phi_q\Big(\int_s^T w(\tau)\nabla\tau + \frac{\sum_{i=1}^{m-2} b_i \int_{\xi_i}^T w(\tau)\nabla\tau}{1-\sum_{i=1}^{m-2} a_i}\Big)\Delta s.
\end{aligned}$$

Therefore, by applying the Arzela-Ascoli theorem on time scales, we can find that $F\overline{K_c}$ is relatively compact.

Finally, from the continuity of f and $w(t) \in C_{ld}([0,T]_{\mathbb{T}}, [0,+\infty))$, we can find that F is continuous. Thus, F is completely continuous. The proof is complete. □

The proof of our main result is based upon an application of the following fixed point theorem in a cone.

Theorem 2.5 ([10])**.** *Let K be a cone in a Banach space X. Let D be an open bounded subset of X with $D_K = D \cap K \neq \emptyset$ and $\overline{D_K} \neq K$. Suppose that F :*

$\overline{D_K} \to K$ is a completely continuous map such that $u \neq Fu$ for $u \in \partial D_K$. Then the following results hold.

(i) *If $\|Fu\| \le \|u\|$ $\forall u \in \partial D_K$, then $i_K(F, D_K) = 1$.*
(ii) *If there exists $e \in K \setminus \{0\}$ such that $u \neq Fu + \lambda e$ $\forall u \in \partial D_K$ and $\lambda > 0$, then $i_K(F, D_K) = 0$.*
(iii) *Let U be open in X such that $\overline{U} \subset D_K$. If $i_K(F, D_K) = 1$ and $i_K(F, U_K) = 0$, then F has a fixed point in $D_K \setminus \overline{U_K}$.*
The same result holds if $i_K(F, D_K) = 0$ and $i_K(F, U_K) = 1$, where $i_K(F, D_K)$ denotes a fixed point index.

3. Main results

Lemma 3.1 ([15])**.** *Let $K_\rho = \{u \in K : \|u\| < \rho\}$ and*

$$\Omega_\rho = \{u \in K : \min_{t \in [0,T]_{\mathbb{T}}} u(t) < \gamma\rho\}.$$

Then Ω_ρ has the following properties:

(a) *Ω_ρ is open relative to K;*
(b) *$K_{\gamma\rho} \subset \Omega_\rho \subset K_\rho$;*
(c) *$u \in \partial\Omega_\rho$ if and only if $\min_{t \in [0,T]_{\mathbb{T}}} u(t) = \gamma\rho$;*
(d) *If $u \in \partial\Omega_\rho$, then $\gamma\rho \le u(t) \le \rho$ for $t \in [0,T]_{\mathbb{T}}$.*

Now, for convenience, we introduce the following notation. Let

$$\frac{1}{\Lambda_1} = \frac{1}{1 - \sum_{i=1}^{m-2} a_i} \int_0^T \phi_q\Big(\int_s^T w(\tau)\nabla\tau + \frac{\sum_{i=1}^{m-2} b_i \int_{\xi_i}^T w(\tau)\nabla\tau}{1 - \sum_{i=1}^{m-2} b_i} \Big) \Delta s,$$

$$\frac{1}{\Lambda_2} = \frac{1}{1 - \sum_{i=1}^{m-2} a_i} \Big[\sum_{i=1}^{m-2} a_i \int_0^{\xi_i} \phi_q\Big(\int_s^T w(\tau)\nabla\tau + \frac{\sum_{i=1}^{m-2} b_i \int_{\xi_i}^T w(\tau)\nabla\tau}{1 - \sum_{i=1}^{m-2} b_i} \Big) \Delta s \Big].$$

We remark that from (H1) it follows that $0 < \Lambda_1, \Lambda_2 < +\infty$, $\Lambda_2\gamma = \Lambda_2\gamma_1\gamma_2 = \Lambda_1\gamma_1 < \Lambda_1$.

Lemma 3.2. *If f satisfies the conditions*

$$f(t,u) \le \phi_p(\Lambda_1\rho) \quad \text{and} \quad u \neq Fu, \text{ for } u \in \partial K_\rho,\ (t,u) \in [0,T]_{\mathbb{T}} \times [0,\rho],$$

then $i_K(F, K_\rho) = 1$.

Proof. If $u \in \partial K_\rho$, then

$$\begin{aligned}
&\int_0^s w(\tau) f^+(\tau, u(\tau))\nabla\tau - \tilde{C} \\
&= \int_0^s w(\tau) f^+(\tau, u(\tau))\nabla\tau \\
&\quad + \frac{\sum_{i=1}^{m-2} b_i \int_0^{\xi_i} w(\tau) f^+(\tau, u(\tau))\nabla\tau - \int_0^T w(\tau) f^+(\tau, u(\tau))\nabla\tau}{1 - \sum_{i=1}^{m-2} b_i} \\
&= \int_0^s w(\tau) f^+(\tau, u(\tau))\nabla\tau \\
&\quad + \frac{\sum_{i=1}^{m-2} b_i \big(\int_0^T w(\tau) f^+(\tau, u(\tau))\nabla\tau + \int_T^{\xi_i} w(\tau) f^+(\tau, u(\tau))\nabla\tau \big)}{1 - \sum_{i=1}^{m-2} b_i}
\end{aligned}$$

$$
\begin{aligned}
&- \frac{\int_0^T w(\tau) f^+(\tau, u(\tau)) \nabla\tau}{1 - \sum_{i=1}^{m-2} b_i} \\
&= \int_0^s w(\tau) f^+(\tau, u(\tau)) \nabla\tau \\
&\quad - \frac{\Big(\int_0^T w(\tau) f^+(\tau, u(\tau)) \nabla\tau - \sum_{i=1}^{m-2} b_i \int_0^T w(\tau) f^+(\tau, u(\tau)) \nabla\tau \Big)}{1 - \sum_{i=1}^{m-2} b_i} \\
&\quad - \frac{\sum_{i=1}^{m-2} b_i \int_{\xi_i}^T w(\tau) f^+(\tau, u(\tau)) \nabla\tau}{1 - \sum_{i=1}^{m-2} b_i} \\
&= \int_0^s w(\tau) f^+(\tau, u(\tau)) \nabla\tau \\
&\quad - \frac{\Big(1 - \sum_{i=1}^{m-2} b_i \Big) \int_0^T w(\tau) f^+(\tau, u(\tau)) \nabla\tau}{1 - \sum_{i=1}^{m-2} b_i} \\
&\quad - \frac{\sum_{i=1}^{m-2} b_i \int_{\xi_i}^T w(\tau) f^+(\tau, u(\tau)) \nabla\tau}{1 - \sum_{i=1}^{m-2} b_i} \\
&= \int_0^s w(\tau) f^+(\tau, u(\tau)) \nabla\tau - \Big(\int_0^T w(\tau) f^+(\tau, u(\tau)) \nabla\tau \\
&\quad + \frac{\sum_{i=1}^{m-2} b_i \int_{\xi_i}^T w(\tau) f^+(\tau, u(\tau)) \nabla\tau}{1 - \sum_{i=1}^{m-2} b_i} \Big) \\
&= - \int_s^T w(\tau) f^+(\tau, u(\tau)) \nabla\tau - \frac{\sum_{i=1}^{m-2} b_i \int_{\xi_i}^T w(\tau) f^+(\tau, u(\tau)) \nabla\tau}{1 - \sum_{i=1}^{m-2} b_i} \\
&\geq -\phi_p(\rho\Lambda_1) \Big[\int_s^T w(\tau) \nabla\tau + \frac{\sum_{i=1}^{m-2} b_i \int_{\xi_i}^T w(\tau) \nabla\tau}{1 - \sum_{i=1}^{m-2} b_i} \Big]
\end{aligned}
$$

so that

$$
\begin{aligned}
\varphi(s) &= \phi_q \Big(\int_0^s w(\tau) f^+(\tau, u(\tau)) \nabla\tau - \tilde{C} \Big) \\
&\geq -\rho\Lambda_1 \phi_q \Big(\int_s^T w(\tau) \nabla\tau + \frac{\sum_{i=1}^{m-2} b_i \int_{\xi_i}^T w(\tau) \nabla\tau}{1 - \sum_{i=1}^{m-2} b_i} \Big)
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
\|Fu\| &\leq \frac{1}{1 - \sum_{i=1}^{m-2} a_i} \Big[\sum_{i=1}^{m-2} a_i \int_{\xi_i}^T \varphi(s) \Delta s - \int_0^T \varphi(s) \Delta s \Big] \\
&\leq - \frac{1}{1 - \sum_{i=1}^{m-2} a_i} \int_0^T \varphi(s) \Delta s \\
&\leq \rho\Lambda_1 \frac{1}{1 - \sum_{i=1}^{m-2} a_i} \int_0^T \phi_q \Big(\int_s^T w(\tau) \nabla\tau + \frac{\sum_{i=1}^{m-2} b_i \int_{\xi_i}^T w(\tau) \nabla\tau}{1 - \sum_{i=1}^{m-2} b_i} \Big) \Delta s \\
&= \rho = \|u\|.
\end{aligned}
$$

This implies $\|Fu\| \le \|u\|$, $u \in \partial K_\rho$. By Theorem 2.5 (i), we have $i_K(F, K_\rho) = 1$. ☐

Lemma 3.3. *If f satisfies the conditions*

$$f(t,u) \ge \phi_p(\Lambda_2\gamma\rho) \quad \text{and} \quad u \ne Fu \text{ for } u \in \partial\Omega_\rho,\ (t,u) \in [0,T]_{\mathbb{T}} \times [\gamma\rho, \rho],$$

then $i_K(F, \Omega_\rho) = 0$.

Proof. Let $e(t) \equiv 1$ for $t \in [0,T]_{\mathbb{T}}$, then $e \in \partial K_1$. We claim that $u \ne Fu + \lambda e$, $u \in \partial\Omega_\rho$, and $\lambda > 0$. In fact, if not, there exist $u_0 \in \partial\Omega_\rho$, and $\lambda_0 > 0$ such that $u_0 = Fu_0 + \lambda_0 e$. Then we find that

$$\begin{aligned}
&\int_0^s w(\tau)f^+(\tau,u_0(\tau))\nabla\tau - \tilde{C}\\
&= \int_0^s w(\tau)f^+(\tau,u_0(\tau))\nabla\tau + \frac{1}{1-\sum_{i=1}^{m-2} b_i}\Big(\sum_{i=1}^{m-2} b_i \int_0^{\xi_i} w(\tau)f^+(\tau,u_0(\tau))\nabla\tau\\
&\quad - \int_0^T w(\tau)f^+(\tau,u_0(\tau))\nabla\tau\Big)\\
&= \int_0^s w(\tau)f^+(\tau,u_0(\tau))\nabla\tau + \frac{\sum_{i=1}^{m-2} b_i}{1-\sum_{i=1}^{m-2} b_i}\Big(\int_0^T w(\tau)f^+(\tau,u_0(\tau))\nabla\tau\\
&\quad + \int_T^{\xi_i} w(\tau)f^+(\tau,u_0(\tau))\nabla\tau\Big) - \frac{\int_0^T w(\tau)f^+(\tau,u_0(\tau))\nabla\tau}{1-\sum_{i=1}^{m-2} b_i}\\
&= \int_0^s w(\tau)f^+(\tau,u_0(\tau))\nabla\tau - \frac{1}{1-\sum_{i=1}^{m-2} b_i}\Big(\int_0^T w(\tau)f^+(\tau,u_0(\tau))\nabla\tau\\
&\quad - \sum_{i=1}^{m-2} b_i \int_0^T w(\tau)f^+(\tau,u_0(\tau))\nabla\tau\Big) - \sum_{i=1}^{m-2} b_i \frac{\int_{\xi_i}^T w(\tau)f^+(\tau,u_0(\tau))\nabla\tau}{1-\sum_{i=1}^{m-2} b_i}\\
&= \int_0^s w(\tau)f^+(\tau,u_0(\tau))\nabla\tau - \frac{\Big(1-\sum_{i=1}^{m-2} b_i\Big)\int_0^T w(\tau)f^+(\tau,u_0(\tau))\nabla\tau}{1-\sum_{i=1}^{m-2} b_i}\\
&\quad - \frac{\sum_{i=1}^{m-2} b_i \int_{\xi_i}^T w(\tau)f^+(\tau,u_0(\tau))\nabla\tau}{1-\sum_{i=1}^{m-2} b_i}\\
&= \int_0^T w(\tau)f^+(\tau,u_0(\tau))\nabla\tau + \int_T^s w(\tau)f^+(\tau,u_0(\tau))\nabla\tau\\
&\quad - \int_0^T w(\tau)f^+(\tau,u_0(\tau))\nabla\tau - \frac{\sum_{i=1}^{m-2} b_i \int_{\xi_i}^T w(\tau)f^+(\tau,u_0(\tau))\nabla\tau}{1-\sum_{i=1}^{m-2} b_i}\\
&= -\int_s^T w(\tau)f^+(\tau,u_0(\tau))\nabla\tau - \frac{\sum_{i=1}^{m-2} b_i \int_{\xi_i}^T w(\tau)f^+(\tau,u_0(\tau))\nabla\tau}{1-\sum_{i=1}^{m-2} b_i}\\
&= -\Big(\int_s^T w(\tau)f^+(\tau,u_0(\tau))\nabla\tau + \frac{\sum_{i=1}^{m-2} b_i \int_{\xi_i}^T w(\tau)f^+(\tau,u_0(\tau))\nabla\tau}{1-\sum_{i=1}^{m-2} b_i}\Big)\\
&\le -\phi_p(\Lambda_2\gamma\rho)\Big(\int_s^T w(\tau)\nabla\tau + \frac{\sum_{i=1}^{m-2} b_i \int_{\xi_i}^T w(\tau)\nabla\tau}{1-\sum_{i=1}^{m-2} b_i}\Big)
\end{aligned}$$

so that

$$\begin{aligned}\tilde{\varphi}(s) &= \phi_q\Big(\int_0^s w(\tau)f^+(\tau,u_0(\tau))\nabla\tau - \tilde{C}\Big)\\ &\le -\Lambda_2\gamma\rho\phi_q\Big(\int_s^T w(\tau)\nabla\tau + \frac{\sum_{i=1}^{m-2} b_i\int_{\xi_i}^T w(\tau)\nabla\tau}{1-\sum_{i=1}^{m-2} b_i}\Big).\end{aligned}$$

From (2.10), we have

$$\begin{aligned}u_0(t) &= Fu_0(t) + \lambda_0 e(t)\\ &\ge \int_0^T \tilde{\varphi}(s)\Delta s + \frac{1}{1-\sum_{i=1}^{m-2} a_i}\Big[\sum_{i=1}^{m-2} a_i\int_{\xi_i}^T \tilde{\varphi}(s)\Delta s - \int_0^T \tilde{\varphi}(s)\Delta s\Big] + \lambda_0\\ &= \frac{-\sum_{i=1}^{m-2} a_i\int_0^T \tilde{\varphi}(s)\Delta s + \sum_{i=1}^{m-2} a_i\int_{\xi_i}^T \tilde{\varphi}(s)\Delta s}{1-\sum_{i=1}^{m-2} a_i} + \lambda_0\\ &= \frac{-\sum_{i=1}^{m-2} a_i\Big(\int_0^{\xi_i} \tilde{\varphi}(s)\Delta s + \int_{\xi_i}^T \tilde{\varphi}(s)\Delta s\Big) + \sum_{i=1}^{m-2} a_i\int_{\xi_i}^T \tilde{\varphi}(s)\Delta s}{1-\sum_{i=1}^{m-2} a_i} + \lambda_0\\ &= \frac{-\sum_{i=1}^{m-2} a_i\int_0^{\xi_i} \tilde{\varphi}(s)\Delta s}{1-\sum_{i=1}^{m-2} a_i} + \lambda_0\\ &\ge \gamma\rho\Lambda_2\frac{\sum_{i=1}^{m-2} a_i\int_0^{\xi_i}}{1-\sum_{i=1}^{m-2} a_i}\phi_q\Big(\int_s^T w(\tau)\nabla\tau + \frac{\sum_{i=1}^{m-2} b_i\int_{\xi_i}^T w(\tau)\nabla\tau}{1-\sum_{i=1}^{m-2} b_i}\Big)\Delta s + \lambda_0\\ &= \gamma\rho + \lambda_0,\end{aligned}$$

which implies $\lambda_0 \le 0$, it is a contradiction. Hence by Theorem 2.5 (ii), it follows that $i_K(F,\Omega_\rho) = 0$. □

Now, we set up and verify our existence results for the BVP (1.1) and (1.2).

Theorem 3.4. *Suppose that one of the following conditions holds.*

- (C1) *There exist ρ_1,ρ_2 and $\rho_3 \in (0,+\infty)$ with $\rho_1 < \gamma\rho_2$, and $\rho_2 < \rho_3$ such that*
  - (i) *$f(t,u) \le \phi_p(\Lambda_1\rho_1)$, $(t,u) \in [0,T]_{\mathbb{T}} \times [0,\rho_1]$;*
  - (ii) *$f(t,u) \ge 0, (t,u) \in [0,T]_{\mathbb{T}} \times [\gamma\rho_1,\rho_3]$, in addition, $f(t,u) \ge \phi_p(\Lambda_2\gamma\rho_2)$, $(t,u) \in [0,T]_{\mathbb{T}} \times [\gamma\rho_2,\rho_2]$, $u \ne Fu, u \in \partial\Omega_{\rho_2}$;*
  - (iii) *$f(t,u) \le \phi_p(\Lambda_1\rho_3)$, $(t,u) \in [0,T]_{\mathbb{T}} \times [0,\rho_3]$.*
- (C2) *There exist ρ_1,ρ_2 and $\rho_3 \in (0,+\infty)$ with $\rho_1 < \rho_2 < \gamma\rho_3$, such that*
  - (i) *$f(t,u) \ge \phi_p(\Lambda_2\gamma\rho_1)$, $(t,u) \in [0,T]_{\mathbb{T}} \times [\gamma^2\rho_1,\rho_2]$;*
  - (ii) *$f(t,u) \le \phi_p(\Lambda_1\rho_2)$, $(t,u) \in [0,T]_{\mathbb{T}} \times [0,\rho_2]$, $u \ne Fu, u \in \partial K_{\rho_2}$;*
  - (iii) *$f(t,u) \ge 0, (t,u) \in [0,T]_{\mathbb{T}} \times [\gamma\rho_2,\rho_3]$, in addition, $f(t,u) \ge \phi_p(\Lambda_2\gamma\rho_3)$, $(t,u) \in [0,T]_{\mathbb{T}} \times [\gamma\rho_3,\rho_3]$.*

Then BVP (1.1) *and* (1.2) *has at least two positive solutions u_1 and u_2.*

Proof. Suppose (C1) holds, we verify that F has a fixed point u_1 either in ∂K_{ρ_1} or u_1 in $\Omega_{\rho_2}\backslash\overline{K_{\rho_1}}$. If $u \ne Fu, u \in \partial K_{\rho_1} \cup \partial K_{\rho_3}$, by Lemmas 3.2 and 3.3, we have

$$i_K(F,K_{\rho_1}) = 1, \quad i_K(F,\Omega_{\rho_2}) = 0, \quad i_K(F,K_{\rho_3}) = 1.$$

By Lemma 3.1 (b) and $\rho_1 < \gamma\rho_2$, we have $\overline{K_{\rho_1}} \subset K_{\gamma\rho_2} \subset \Omega_{\rho_2}$. From Theorem 2.5 (iii), we obtain F has a fixed point $u_1 \in \Omega_{\rho_2}\backslash\overline{K_{\rho_1}}$. Similarly, F has a fixed point

$u_2 \in K_{\rho_3}\backslash\overline{\Omega_{\rho_2}}$. Obviously,

$$\|u_1\| > \rho_1, \quad \min_{t\in[0,T]_{\mathbb{T}}} u_1(t) = u_1(0) \geq \gamma\|u_1\| > \gamma\rho_1.$$

Therefore, $\gamma\rho_1 \leq u_1(t) \leq \rho_2$, $t \in [0,T]_{\mathbb{T}}$. By (C1) (ii), we have $f(t,u_1(t)) \geq 0$, $t \in [0,T]_{\mathbb{T}}$, i.e. $f^+(t,u_1(t)) = f(t,u_1(t))$. Hence, $Fu_1 = Au_1$. Consequently u_1 is a fixed point of A. From $u_2 \in K_{\rho_3}\backslash\overline{\Omega_{\rho_2}}$, $\rho_2 < \rho_3$ and Lemma 3.1 (b), we have $K_{\gamma\rho_2} \subset \Omega_{\rho_2} \subset K_{\rho_3}$. Clearly, $\|u_2\| > \gamma\rho_2$. This implies

$$\min_{t\in[0,T]_{\mathbb{T}}} u_2(t) = u_2(0) \geq \gamma\|u_2\| > \gamma^2\rho_2.$$

So,

$$\gamma^2\rho_2 \leq u_2(t) \leq \rho_3, \quad t \in [0,T]_{\mathbb{T}}.$$

Using that $\rho_1 < \gamma\rho_2$ and (C1) (ii), we obtain $f(t,u_2(t)) \geq 0$, $t \in [0,T]_{\mathbb{T}}$, i.e., $f^+(t,u_2(t)) = f(t,u_2(t))$. Therefore u_2 is another fixed point of A. Thus, we have verified that BVP (1.1) and (1.2) has at least two positive solutions u_1 and u_2.

The proof is similar when (C2) holds. The proof is complete. □

By a similar argument to that of Theorem 3.4, we can find the following new results on existence of at least one positive solution of the BVP (1.1) and (1.2).

Theorem 3.5. *Assume that one of the following conditions holds.*

- (C3) *There exist* $\rho_1, \rho_2 \in (0,+\infty)$ *with* $\rho_1 < \gamma\rho_2$ *such that*
  - (i) $f(t,u) \leq \phi_p(\Lambda_1\rho_1), (t,u) \in [0,T]_{\mathbb{T}} \times [0,\rho_1]$;
  - (ii) $f(t,u) \geq 0, (t,u) \in [0,T]_{\mathbb{T}} \times [\gamma\rho_1,\rho_2]$, *in addition,* $f(t,u) \geq \phi_p(\Lambda_2\gamma\rho_2)$, $(t,u) \in [0,T]_{\mathbb{T}} \times [\gamma\rho_2,\rho_2]$.
- (C4) *There exist* $\rho_1, \rho_2 \in (0,+\infty)$ *with* $\rho_1 < \rho_2$ *such that*
  - (i) $f(t,u) \geq \phi_p(\Lambda_2\gamma\rho_1)$, $(t,u) \in [0,T]_{\mathbb{T}} \times [\gamma^2\rho_1,\rho_2]$;
  - (ii) $f(t,u) \leq \phi_p(\Lambda_1\rho_2)$, $(t,u) \in [0,T]_{\mathbb{T}} \times [0,\rho_2]$.

Then BVP (1.1) *and* (1.2) *has at least one positive solution.*

4. Application

Let $\mathbb{T} = \{1-(\frac{1}{2})^{\mathbb{N}_0}\} \cup \{\frac{1}{3},1\}$, $\mathbb{N}_0$ denotes the set of all nonnegative integers. We consider the following BVP on time scales

$$(u^{\Delta}(t))^{\nabla} + f(t,u(t)) = 0, \quad t \in [0,T]_{\mathbb{T}}, \tag{4.1}$$

$$u(0) = \frac{2}{3}u\Big(\frac{1}{3}\Big) + \frac{1}{4}u\Big(\frac{1}{2}\Big), \quad u^{\Delta}(1) = \frac{1}{3}u^{\Delta}\Big(\frac{1}{3}\Big) + \frac{1}{2}u\Big(\frac{1}{2}\Big), \tag{4.2}$$

where

$$f(t,u) = \begin{cases} \big(u-\frac{5455}{41472}\big)^3 \times \frac{1}{40}t^3, & (t,u) \in [0,T]_{\mathbb{T}} \times [0,\frac{5455}{41472}], \\ \frac{1}{40}t^3 \sin\big(\frac{41472}{36017}\frac{\pi}{2}u - \frac{5455}{36017}\frac{\pi}{2}\big), & t,u) \in [0,T]_{\mathbb{T}} \times [\frac{5455}{41472},1], \\ \frac{1}{40}t^3\big[\frac{54550}{3078} - \frac{41472}{3078}u\big] + \frac{25}{288}\big[\frac{41472}{3078}u - \frac{41472}{3078}\big], & (t,u) \in [0,T]_{\mathbb{T}} \times [1,\frac{54550}{41472}], \\ \frac{25}{288} + \frac{7}{7843}t^3\big(u-\frac{54550}{41472}\big)^2, & (t,u) \in [0,T]_{\mathbb{T}} \times [\frac{54550}{41472},10], \\ \frac{25}{288} + \frac{7}{7843}t^3\big(10-\frac{54550}{41472}\big)^2[1+(u-10)(20-u)], & (t,u) \in [0,T]_{\mathbb{T}} \times [10,+\infty). \end{cases}$$

It is easy to see that $f:[0,1]\times[0,+\infty)\to(-\infty,+\infty)$ is continuous. In this case $p=q=2, w(t)\equiv 1$, $T=1$, $a_1=\frac{2}{3}$, $a_2=\frac{1}{4}$, $b_1=\frac{1}{3}$, $b_2=\frac{1}{2}$, $\xi_1=\frac{1}{3}$, $\xi_2=\frac{1}{2}$, it follows from a direct calculation that

$$
\begin{aligned}
\frac{1}{\Lambda_1}&=\frac{1}{1-(a_1+a_2)}\int_0^T\phi_q\Big(\int_s^T\nabla\tau+\frac{b_1\int_{\xi_1}^T\nabla\tau+b_2\int_{\xi_2}^T\nabla\tau}{1-(b_1+b_2)}\Big)\Delta s\\
&=\frac{1}{1-(\frac{2}{3}+\frac{1}{4})}\int_0^1\Big(1-s+\frac{\frac{1}{3}\Big(1-\frac{1}{3}\Big)+\frac{1}{2}\Big(1-\frac{1}{3}\Big)}{1-(1-\frac{1}{3}+\frac{1}{2})}\Big)\Delta s=40,\\
\frac{1}{\Lambda_2}&=\frac{1}{1-(a_1+a_2)}\Big[a_1\int_0^{\xi_1}\phi_q\Big(\int_s^T\nabla\tau+\frac{b_1\int_{\xi_1}^T\nabla\tau+b_2\int_{\xi_2}^T\nabla\tau}{1-(b_1+b_2)}\Big)\Delta s\\
&\quad+a_2\int_0^{\xi_2}\phi_q\Big(\int_s^T\nabla\tau+\frac{b_1\int_{\xi_1}^T\nabla\tau+b_2\int_{\xi_2}^T\nabla\tau}{1-(b_1+b_2)}\Big)\Delta s\Big]\\
&=\frac{1}{1-(\frac{2}{3}+\frac{1}{4})}\Big[\frac{2}{3}\int_0^{1/3}\Big(1-s+\frac{\frac{1}{3}\Big(1-\frac{1}{3}\Big)+\frac{1}{2}\Big(1-\frac{1}{2}\Big)}{1-(\frac{1}{3}+\frac{1}{2})}\Big)\Delta s\\
&\quad+\frac{1}{4}\int_0^{1/2}\Big(1-s+\frac{\frac{1}{3}\Big(1-\frac{1}{3}\Big)+\frac{1}{2}\Big(1-\frac{1}{2}\Big)}{1-\Big(\frac{1}{3}+\frac{1}{2}\Big)}\Big)\Delta s\Big]=\frac{1091}{72},\\
\gamma_1&=a_1\frac{\xi_1}{T}+a_2\frac{\xi_2}{T}=\frac{2}{3}\cdot\frac{1}{3}+\frac{1}{4}\cdot\frac{1}{2}=\frac{25}{72}.
\end{aligned}
$$

Let

$$
\begin{aligned}
N&=a_1\int_0^{\xi_1}\phi_q\Big(\int_s^T\nabla\tau+\frac{b_1\int_{\xi_1}^T\nabla\tau+b_2\int_{\xi_2}^T\nabla\tau}{1-(b_1+b_2)}\Big)\Delta s\\
&\quad+a_2\int_0^{\xi_2}\phi_q\Big(\int_s^T\nabla\tau+\frac{b_1\int_{\xi_1}^T\nabla\tau+b_2\int_{\xi_2}^T\nabla\tau}{1-(b_1+b_2)}\Big)\Delta s\\
&=\frac{2}{3}\int_0^{1/3}\Big(1-s+\frac{\frac{1}{3}\Big(1-\frac{1}{3}\Big)+\frac{1}{2}\Big(1-\frac{1}{2}\Big)}{1-\Big(\frac{1}{3}+\frac{1}{2}\Big)}\Big)\Delta s\\
&\quad+\frac{1}{4}\int_0^{1/2}\Big(1-s+\frac{\frac{1}{3}\Big(1-\frac{1}{3}\Big)+\frac{1}{2}\Big(1-\frac{1}{2}\Big)}{1-\Big(\frac{1}{3}+\frac{1}{2}\Big)}\Big)\Delta s=\frac{1091}{864}
\end{aligned}
$$

and

$$
\begin{aligned}
D&=\int_0^T\phi_q\Big(\int_s^T\nabla\tau+\frac{b_1\int_{\xi_1}^T\nabla\tau+b_2\int_{\xi_2}^T\nabla\tau}{1-(b_1+b_2)}\Big)\Delta s\\
&=\int_0^1\Big(1-s+\frac{\frac{1}{3}\Big(1-\frac{1}{3}\Big)+\frac{1}{2}\Big(1-\frac{1}{3}\Big)}{1-\Big(\frac{1}{3}+\frac{1}{2}\Big)}\Big)\Delta s=\frac{10}{3}.
\end{aligned}
$$

Then

$$
\gamma_2=\frac{N}{D}=\frac{1091}{864}\cdot\frac{3}{10}=\frac{1091}{2880},\quad \gamma=\gamma_1\gamma_2=\frac{5455}{41472},
$$

$$\Lambda_2\gamma = \frac{72}{1091}\cdot\frac{5455}{41472} = \frac{24}{2765} \approx 0.00868 < \frac{1}{40} = 0.025 = \Lambda_1.$$

Choose $\rho_1 = 1$, $\rho_2 = 10$, $\rho_3 = 20$, it is easy to check that $\gamma\rho_1 < \rho_1 < \gamma\rho_2 < \rho_2 < \rho_3$. By (2.10), we obtain

$$\begin{aligned}
(Fu)(t) &= \int_t^T \phi_q\Big(\int_0^s f^+(\tau,u(\tau))\nabla\tau - \tilde{C}\Big)\Delta s \\
&\quad + \frac{1}{1-\sum_{i=1}^{m-2} a_i}\Big[\sum_{i=1}^{m-2} a_i \int_{\xi_i}^T \phi_q\Big(\int_0^s f^+(\tau,u(\tau))\nabla\tau - \tilde{C}\Big)\Delta s \\
&\quad - \int_0^T \phi_q\Big(\int_0^s f^+(\tau,u(\tau))\nabla\tau - \tilde{C}\Big)\Delta s\Big] \\
&= \int_t^1 \Big(\int_0^s f^+(\tau,u(\tau))\nabla\tau - \tilde{C}\Big)\Delta s + 8\int_{1/3}^1 \Big(\int_0^s f^+(\tau,u(\tau))\nabla\tau - \tilde{C}\Big)\Delta s \\
&\quad + 3\int_{1/2}^1 \Big(\int_0^s f^+(\tau,u(\tau))\nabla\tau - \tilde{C}\Big)\Delta s \\
&\quad - 12\int_0^1 \Big(\int_0^s f^+(\tau,u(\tau))\nabla\tau - \tilde{C}\Big)\Delta s + \frac{31}{6}\tilde{C},
\end{aligned}$$

where

$$\begin{aligned}
\tilde{C} &= \frac{-1}{1-\sum_{i=1}^{m-2} b_i}\Big[\sum_{i=1}^{m-2} b_i \int_0^{\xi_i} f^+(\tau,u(\tau))\nabla\tau - \int_0^T f^+(\tau,u(\tau))\nabla\tau\Big] \\
&= -2\int_0^{1/3} f^+(\tau,u(\tau))\nabla\tau - 3\int_0^{1/2} f^+(\tau,u(\tau))\nabla\tau + 6\int_0^1 f^+(\tau,u(\tau))\nabla\tau.
\end{aligned}$$

Since $f(t,u) \le \frac{13}{80}$, $t \in [0,T]_{\mathbb{T}}$, $u \in [0,10]$, for $u \in \partial K_{10}$, we have

$$\begin{aligned}
\|Fu\| &= \max_{t\in[0,T]_{\mathbb{T}}} Fu(t) \\
&= 8\int_{1/3}^1 \Big(\int_0^s f^+(\tau,u(\tau))\nabla\tau\Big)\Delta s + 3\int_{1/2}^1 \Big(\int_0^s f^+(\tau,u(\tau))\nabla\tau\Big)\Delta s \\
&\quad - 12\int_0^1 \Big(\int_0^s f^+(\tau,u(\tau))\nabla\tau\Big)\Delta s + \frac{31}{6}\tilde{C} \\
&= 8\int_{1/3}^1 \Big(\int_0^s f^+(\tau,u(\tau))\nabla\tau\Big)\Delta s + 3\int_{1/2}^1 \Big(\int_0^s f^+(\tau,u(\tau))\nabla\tau\Big)\Delta s \\
&\quad - 12\int_0^1 \Big(\int_0^s f^+(\tau,u(\tau))\nabla\tau\Big)\Delta s + \frac{31}{6}\Big[-2\int_0^{1/3} f^+(\tau,u(\tau))\nabla\tau \\
&\quad - 3\int_0^{1/2} f^+(\tau,u(\tau))\nabla\tau + 6\int_0^1 f^+(\tau,u(\tau))\nabla\tau\Big] \\
&\le \frac{13}{4} < 10 = \|u\|.
\end{aligned}$$

This implies $Fu \ne u$, for $u \in \partial K_{10}$.

As a result, f satisfies the following conditions:

(i) $f(t,u) \le \phi_p(\Lambda_1\rho_1) = \frac{1}{40}$, $(t,u) \in [0,T]_{\mathbb{T}} \times [0,1]$;

(ii) $f(t,u) \geq 0$, $(t,u) \in [0,T]_{\mathbb{T}} \times [\frac{5455}{41472}, 20]$; in addition, $f(t,u) \geq \phi_p(\Lambda_2 \gamma \rho_2) = \frac{72}{1091} \cdot \frac{5455}{41472} \cdot 4 = \frac{5}{144}$, $(t,u) \in [0,T]_{\mathbb{T}} \times [\frac{54550}{41472}, 10]$;

(iii) $f(t,u) \leq \phi_p(\Lambda_1 \rho_3) = \frac{7}{40}$, $(t,u) \in [0,T]_{\mathbb{T}} \times [0, 20]$.

Therefore, (C1) of Theorem 3.4 is satisfied, then problem (4.1) and (4.2) has two positive solutions u_1, u_2 satisfying

$$\|u_1\| \leq 10, \quad \|u_2\| > 10.$$

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Abdulkadir Dogan
Department of Applied Mathematics, Faculty of Computer Sciences, Abdullah Gul University, Kayseri, 38039 Turkey.
Tel +90 352 224 88 00, Fax +90 352 338 88 28
E-mail address: abdulkadir.dogan@agu.edu.tr