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Interpolated variational iteration method for solving the jamming transition problem

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Abstract

The purpose of this study is to present an analytical based numerical solution for Jamming Transition Problem (JTP) using Interpolated Variational Iteration Method (IVIM). The method eliminates the difficulties on analytical integration of expressions in analytical variational iteration technique and provides numerical results with analytical accuracy. JTP may be transformed into a nonlinear non-conservative oscillator by Lorenz system in which jamming transition is presented as spontaneous deviations of headway and velocity caused by the acceleration/breaking rate to be higher than the critical value. The resulting governing equation of JTP has no exact solution due to existing nonlinearities in the equation. The problem was previously attempted to be solved semi-analytically via analytical approximation methods including analytical variational iteration technique. The results of this study show that IVIM solutions agree very well with the numerical solution provided by the mathematical software. IVIM with two different formulation according to governing equation is introduced. Required order of the solution and number of time steps for a good agreement is determined according to the analyses performed using IVIM.

Keywords: Analytical approximate solution; Interpolated Variational Iteration Method; Jamming Transition Problem; Lorenz system

1. Introduction

The traffic flow is a major interest of practical transportation applications based on the models introducing the features of moving objects. Early on modeling of traffic flow was based on first order conservation laws of fluids [33,44]. Prigogine [42] developed kinetic theory for traffic flow. Payne [40] and Whitham [58] derived second order approximations of car-following equations. Daganzo [7] examined the flaws in higher order traffic flow models. Zhang [63] proposed an improved non-equilibrium model by replacing traffic sound speed in PW model [40,58] by a concentration dependent term. Aw and Rascle [2] developed a second order traffic model resolving nonphysical effects existing in previous models. Models attempting to describe traffic flow phenomena on the basis of single vehicle elements are called microscopic models while macroscopic models explain the traffic phenomena are with collective traffic properties [31]. Kinetic models are based on the analogy to gas dynamics that lead to Boltzmann

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type equations [55]. Kinetic models are suitable for the flows with lower or medium traffic densities while at higher densities continuum models perform better [31]. For a continuous traffic flow, it is important to determine the behavior of transported entities that may cause traffic congestion which occurs when the volume of traffic flow exceeds the capacity of a node in a highway network where bottleneck occurs at these nodes. Many studies exist related to traffic congestion. [4,49,64,61,3,52,5,50,57], microscopic models [6,9,32], macroscopic models [39,30,29,23,54,22,62,48,8,28] and kinetic models [42,43,41,36,26,27,10] in literature. Some researchers focused on the jamming transition problem (JTP) in various forms such as thermodynamic, hydrodynamic and kinetic theories [34,14,15], car-following model [13,37,51,47] and cellular automaton model [35,60,53].

JTP is transformed into a nonlinear non-conservative oscillatory system with a restoring damping term, via the Lorenz system [38,24,25]. The governing equation of the JTP is a highly nonlinear differential equation. The equation was solved with different analytical approximate solution techniques such as the Homotopy Perturbation Method (HPM) and the Variational Iteration Method (VIM) [12], the Differential Transformation Method (DTM) [11], the Homotopy Analysis Method (HAM) [16] and the Adomian Decomposition Method (ADM) [46].

In this study, the governing equation of JTP is solved using Interpolated Variational Iteration Method (IVIM) introducing two different formulations due to Lagrangian multiplier described in variational iteration method. VIM is an analytical technique developed by He [17]. VIM has been successfully used in the solution of linear/nonlinear ordinary/partial differential equations [18,19,21,56] and has great potential to be discovered [20,59]. A numerical interpretation of VIM for initial value problems was derived by Salkuyeh and Tavakoli [45] named as IVIM. Later, Atay et al. [1] applied IVIM for the solution of stiff differential equations. IVIM can be described as discretized formulation of the VIM. Although the analytical derivation of the solution brings the expectation of more accurate results, analytical integration process sometimes requires enormous time durations depending on the functions to be integrated. IVIM reduces the integration time due to nonlinear terms existing in the equation that is a great advantage over analytical approximation techniques such as VIM, HPM, DTM, ADM and HAM. This study investigates the application of IVIM first time to the presented problem.

2. Governing JTP equation

JTP is converted to a nonlinear non-conservative oscillator by Lorenz system that presents the jamming transition as spontaneous deviations of headway and velocity caused by the acceleration/breaking rate to be higher than the critical value. The complex dynamics of traffic flow in the one-lane highway can be defined using Lorenz scheme based on the car following model for which the formulation process is summarized below [38,24,25].

Suppose that Δx is the headway between two observed vehicles. The absolute value of headway deviation is introduced as

$$\eta \equiv |\Delta x - h| \tag{1}$$

where h is optimal headway or safety distance. The velocity deviation due to variation of headway deviation from optimal value $v_0 - V$ is

$$v \equiv \Delta\dot{x} - (v_0 - V) \tag{2}$$

where $v_0 = h/t_0$. Here, t_0 is nominal time lag, v_0 is optimal speed vehicle, V is actual velocity of observed vehicle and x is vehicle coordinate.

Khomenko et al. [25] reported that in the equations of motion it is assumed that in the autonomous mode headway deviation η, headway deviation velocity v and acceleration/braking time τ have dissipative behavior and their relaxation to equilibrium state is governed by Debye type equations with corresponding relaxation times t_η, t_v, and t_τ.

Besides headway deviation η and its velocity deviation v should vary to prevent the growth of τ, since the decrease of acceleration/breaking time support to the formation of stable traffic flow according to LeChatelier principle. These issues are taken into account by the Lorenz system that describes the self-organization process.

Time rate of change of headway deviation is defined as the rate of change of headway deviation to its stationary position.

$$\dot{\eta} = \frac{0 - \eta}{t_\eta} + v = -\frac{\eta}{t_\eta} + v \tag{3}$$

The rate of velocity deviation variation and the rate of acceleration/breaking time are supposed to have the following forms.

$$\dot{v} = \frac{0-v}{t_v} + g_v \eta \tau = -\frac{v}{t_v} + g_v \eta \tau \tag{4}$$

$$\dot{\tau} = \frac{\tau_0 - \tau}{t_\tau} - g_\tau \eta v \tag{5}$$

In Eqs. (4)–(5), t_v, t_τ, g_v and g_τ are positive constants and dot mean differentiation with respect to time. In Eq. (5) the stationary accelerating/braking time τ_0 required to take characteristic velocity is assumed as a finite value.

In the set of equations, the fluctuation of acceleration/braking time is not considered. In Eqs. (3)–(5), the first terms on the right-hand side of equations characterize the relaxation of each quantity to an equilibrium value and the second term on the right-hand side of Eq. (4) provides a positive feedback of headway deviation η and acceleration/breaking time τ on velocity deviation ν. The velocity deviation is increased due to the positive feedback that leads to traffic jam formation.

There is no available analytical solution for the set of equations (3)–(5) and some simple assumptions introduced to obtain an admissible solution such that $t_\eta \gg t_\tau$ and $t_\eta \approx t_\nu$, the relaxation time of acceleration/breaking is too small when compared with the relaxation times for headway deviation and velocity deviation. According to these conditions, coordination of acceleration/breaking time τ occurs through the variation of headway and velocity deviations. The assumptions lead to the small parameter $t_\tau \dot{\tau} \approx 0$ and Eq. (5) turns into

$$\tau = \tau_0 - g_\tau t_\tau \eta v \tag{6}$$

In order to form a one-parameter model, velocity deviation and time rate of velocity deviation in Eq. (4) are determined from Eq. (3) are substituted in Eq. (4) simultaneously with Eq. (6).

$$\ddot{\eta} + \frac{\dot{\eta}}{t_\eta} = -\frac{1}{t_v}\left(\dot{\eta} + \frac{\eta}{t_\eta}\right) + g_v \eta \left(\tau_0 - g_\tau t_\tau \eta v\right) \tag{7}$$

The velocity deviation in Eq. (6) now appears in Eq. (7) and it is again substituted in the expression determined from Eq. (3).

$$\ddot{\eta} + \frac{\dot{\eta}}{t_\eta} = -\frac{1}{t_v}\left(\dot{\eta} + \frac{\eta}{t_\eta}\right) + g_v \eta \left[\tau_0 - g_\tau t_\tau \eta \left(\dot{\eta} + \frac{\eta}{t_\eta}\right)\right] \tag{8}$$

Eq. (8) can be rearranged as

$$\ddot{\eta} + \frac{\dot{\eta}}{t_\eta}\left(1 + \frac{t_\eta}{t_v} + g_v g_\tau t_\eta t_\tau \eta^2\right) = \eta\left(g_v \tau_0 - \frac{1}{t_v t_\eta}\right) - \frac{g_v g_\tau t_\tau}{t_\eta}\eta^3 \tag{9}$$

Multiplying both sides of Eq. (9) by t_η^2 produces the following equation.

$$t_\eta^2 \ddot{\eta} + t_\eta \dot{\eta}\left(1 + \frac{t_\eta}{t_v} + g_v g_\tau t_\eta t_\tau \eta^2\right) = \eta\left(g_v t_\eta^2 \tau_0 - \frac{t_\eta}{t_v}\right) - g_v g_\tau t_\eta t_\tau \eta^3 \tag{10}$$

Introducing natural scale factors t_η, $\eta_m = \left(g_v g_\tau t_\eta t_\tau\right)^{-1/2}$, $v_m = \eta_m / t_\eta = t_\eta^{-3/2}\left(g_v g_\tau t_\tau\right)^{-1/2}$, $\tau_c = \left(g_v t_\eta^2\right)^{-1}$ into Eq. (10) for time, headway deviation, velocity deviation and acceleration/braking time respectively give the following form.

$$t_\eta^2 \ddot{\eta} + t_\eta \dot{\eta}\left(1 + \frac{t_\eta}{t_v} + \frac{\eta^2}{\eta_m^2}\right) = \eta\left(\frac{\tau_0}{\tau_c} - \frac{t_\eta}{t_v}\right) - \frac{\eta^3}{\eta_m^2} \tag{11}$$

Dividing both sides of Eq. (11) by η_m constitutes following normalized form.

$$\frac{\ddot{\eta}}{\eta_m / t_\eta^2} + \frac{\dot{\eta}}{\eta_m / t_\eta}\left(1 + \frac{t_\eta}{t_v} + \frac{\eta^2}{\eta_m^2}\right) = \frac{\eta}{\eta_m}\left(\frac{\tau_0}{\tau_c} - \frac{t_\eta}{t_v}\right) - \frac{\eta^3}{\eta_m^3} \tag{12}$$

Introducing nondimensional constants $\sigma \equiv t_\eta / t_v$ and $\varepsilon \equiv \tau_0 / \tau_c$ and redefining η as normalized headway deviation, following nondimensional governing equation for one-parameter model is developed for JTP.

$$\ddot{\eta} + \dot{\eta}\left(1 + \sigma + \eta^2\right) - \eta\left(\varepsilon - \sigma\right) + \eta^3 = 0 \tag{13}$$

Eq. (13) is a third order nonlinear differential equation possessing no analytical solution due to highly nonlinear terms existing in the equation. In such problems, numerical, variational or analytical approximate solution techniques may be employed to provide a solution.

3. Interpolated variational iteration method (IVIM)

Salkuyeh and Tavakoli [45] developed IVIM to solve a one-dimensional initial value problem.

$$u'(t) = f(t, u(t))\, u(a) = u_a t \in [a, T] \tag{14}$$

VIM formulation for the problem given in Eq. (14) may be written according to He [17] as follows:

$$u_{m+1}(t) = u_m(t) + \int_a^t \lambda(\xi, t)\left(u_m'(\xi) - f(\xi, u_m(\xi))\right) d\xi \tag{15}$$

Salkuyeh and Tavakoli [45] applied integrating by parts to Eq. (15) assuming $u_0(a) = 0$ to obtain following procedure.

$$u_{m+1}(t) = G_m(t) - \int_a^t H_m(\xi, t)\, d\xi \tag{16}$$

where

$$G_m(t) = (1 + \lambda(t, t))\, u_m(t) - \lambda(a, t)\, u_m(a) \tag{17}$$

$$H_m(\xi, t) = \frac{d\lambda(\xi, t)}{d\xi} u_m(\xi) + \lambda(\xi, t)\, f(\xi, u_m(\xi)) \tag{18}$$

IVIM was applied to Eq. (16) dividing the interval $[a, T]$ into $n-1$ subintervals, which discretized the solution domain with the following nodes.

$$t_i = a + (i-1)\, h, \quad i = 1, 2, \ldots, n \quad h = \frac{T-a}{n-1} \tag{19}$$

At this step, Salkuyeh and Tavakoli [45] defined B-spline basis function of first order at the nodes and a numerical integration procedure was obtained applying a piecewise linear interpolation to $H_m(\xi, t)$ in Eq. (16). The method leads to the following numerical equivalent of VIM formulation in Eq. (15).

$$u_{m+1}(t_i) \approx \hat{u}_{m+1}(t_i) = G_m(t_i) - h \sum_{r=2}^{i-1} H_m(t_r, t_i) - \frac{h}{2} H_m(t_i, t_i) \tag{20}$$

where index $m+1$ denotes $m+1$st order solution.

4. IVIM for JTP

JTP was previously solved with different analytical approximation techniques [12,11,46] with the following initial conditions.

$$\eta(0) = A \qquad \dot{\eta}(0) = 0 \tag{21}$$

JTP is governed by a system of two first order equations based on Eq. (13) as follows:

$$\dot{\eta} - \chi = 0, \qquad \eta(0) = A \tag{22}$$

$$\dot{\chi} + \chi\left(1 + \sigma + \eta^2\right) - \eta(\varepsilon - \sigma) + \eta^3 = 0, \qquad \chi(0) = 0 \tag{23}$$

Lagrangian multiplier for Eq. (22) can be determined by imposing the variation and by considering the restricted variation which reduces

$$\delta\eta_{m+1}(t) = \delta\eta_m(t) + \delta \int_0^t \lambda(\xi, t)\, \dot{\eta}_m(\xi)\, d\xi \tag{24}$$

$$\delta\eta_{m+1}(t) = \delta\eta_m(t) + \lambda(\xi, t)\, \delta\eta_m(\xi)\big|_{\xi=t} - \int_0^t \lambda'(\xi, t)\left(\int_0^\xi \delta\dot{\eta}_m(\tau)\, d\tau\right) d\xi \tag{25}$$

Thus, the equations below are obtained.

$$\lambda' (\xi, t) = 0. \qquad 1 + \lambda (\xi, t)|_{\xi=t} = 0 \tag{26}$$

Hence, Lagrangian multiplier for Eq. (22) becomes

$$\lambda_\eta (\xi, t) = -1 \tag{27}$$

However, two different Lagrangian multipliers may be used for Eq. (23) by assuming linear operator in two different ways.

4.1. Formulation I

Assuming $L\chi = \dot{\chi}$ in Eq. (23) Lagrangian multiplier can be determined similarly like performed between Eqs. (24)–(27) with $L\eta = \dot{\eta}$. Thus,

$$\lambda_\chi (\xi, t) = -1 \tag{28}$$

Then, IVIM for JTP would be obtained as follows:

$$G_m^\chi (t) = 0 \tag{29}$$

$$H_m^\chi (\xi, t) = \chi_m (\xi) \left(1 + \sigma + \eta_m (\xi)^2\right) + \eta(\xi) (\varepsilon - \sigma) - \eta(\xi)^3 \tag{30}$$

$$G_m^\eta (t) = A \tag{31}$$

$$H_m^\eta (\xi, t) = \chi_m (\xi) \tag{32}$$

$$\hat{\chi}_{m+1} (t_i) = G_m^\chi (t_i) - h \sum_{r=2}^{i-1} H_m^\chi (t_r, t_i) - \frac{h}{2} H_m^\chi (t_i, t_i) \tag{33}$$

$$\hat{\eta}_{m+1} (t_i) = G_m^\eta (t_i) - h \sum_{r=2}^{i-1} H_m^\eta (t_r, t_i) - \frac{h}{2} H_m^\eta (t_i, t_i) \tag{34}$$

4.2. Formulation II

Assuming $L\chi = \dot{\chi} + (1 + \sigma)\chi$ in Eq. (23) variational procedure would be maintained as follows:

$$\delta\chi_{m+1} (t) = \delta\chi_m (t) + \delta \int_0^t \lambda (\xi, t) [\dot{\chi}_m (\xi) + (1 + \sigma)\chi_m (\xi)] \, d\xi \tag{35}$$

$$\begin{aligned}\delta\chi_{m+1} (t) = \delta\chi_m (t) + \lambda (\xi, t) \, \delta\chi_m (\xi)|_{\xi=t} - \int_a^t \lambda' (\xi, t) \left(\int_0^\xi \delta\dot{\chi}_m (\tau) \, d\tau\right) d\xi \\ + \int_0^t \lambda (\xi, t) (1 + \sigma) \, \delta\chi_m(\xi) d\xi\end{aligned} \tag{36}$$

$$\delta\chi_{m+1} (t) = \delta\chi_m (t) + \lambda (\xi, t) \, \delta\chi_m (\xi)|_{\xi=t} - \int_a^t \left[\lambda' (\xi, t) - (1 + \sigma) \lambda (\xi, t)\right] \delta\chi_m(\xi) d\xi \tag{37}$$

Thus, the following equations are obtained.

$$\lambda' (\xi, t) - (1 + \sigma) \lambda (\xi, t) = 0. \qquad 1 + \lambda (\xi, t)|_{\xi=t} = 0 \tag{38}$$

Hence, Lagrangian multiplier for Eq. (23) may be determined as

$$\lambda_\chi (\xi, t) = -e^{(1+\sigma)(\xi - t)} \tag{39}$$

Then, IVIM for JTP would be obtained as follows:

$$G_m^\chi (t) = 0 \tag{40}$$

$$H_m^\chi (\xi, t) = e^{(1+\sigma)(\xi - t)} \left(\chi_m (\xi) \, \eta_m (\xi)^2 - \eta (\xi) (\varepsilon - \sigma) + \eta(\xi)^3\right) \tag{41}$$

$$G_m^\eta (t) = A \tag{42}$$

$$H_m^\eta (\xi, t) = \chi_m (\xi) \tag{43}$$

$\hat{\chi}_{m+1} (t_i)$ and $\hat{\eta}_{m+1} (t_i)$ can be obtained using Eqs. (30)–(31).

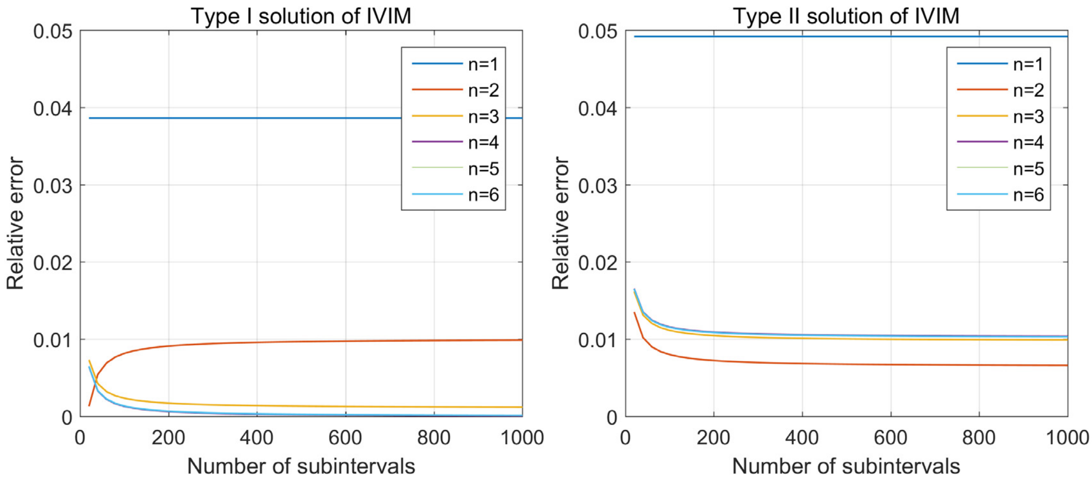

Fig. 1. The relative errors between IVIM and numerical solutions with Type I and Type II solutions of the IVIM for Mode 1.

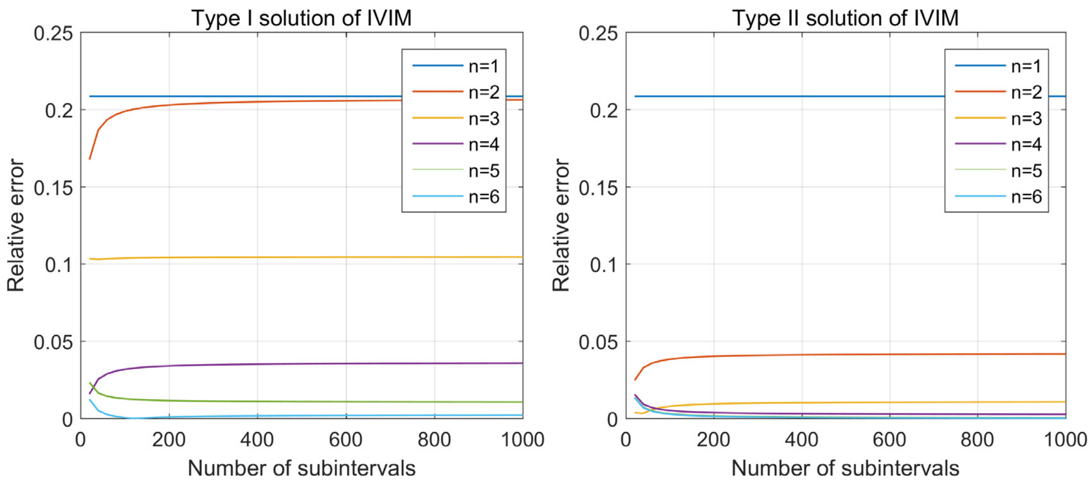

Fig. 2. The relative errors between IVIM and numerical solutions with Type I and Type II solutions of the IVIM for Mode 2.

5. Results

In order to investigate the behavior of the IVIM solution, we select four different modes as different values of the parameters in the governing JTP equation. The values of parameters for different modes are shown in Table 1.

The order of solution and number of subintervals in IVIM directly affect the accuracy and elapsed time of the solution. To define the optimum order and subintervals, relative error values between IVIM and numerical solutions have been tested with first to sixth order solutions and from 20 to 1000 subintervals for 4 modes. The results are shown in Figs. 1–4. The figures show that a minimum of fifth or sixth order solution with 200 or more subintervals is generally sufficient for an accurate solution. The relative error between IVIM and numerical solution is calculated using Eq. (44).

$$Relative\ Error = |IVIM\text{-}numerical| \,/\, |numerical| \tag{44}$$

IVIM results either of type 1 or type 2 were obtained from a MATLAB program coded by authors. Numerical values were calculated using NSolve command in Wolfram Mathematica Software. Solutions for each mode are

Table 1
Different values for parameters ε, σ and t.

Mode	ε	σ	t
1	0.25	0.75	0.25
2	0.75	2.50	0.50
3	3.25	0.75	0.75
4	2.00	0.75	1.00

Table 2
Results of numerical, IVIM Type I and IVIM Type II solutions for different values of initial headways in Mode 1.

IVIM conditions		Type of solution	Initial headway						
Order	Subintervals		0.1	0.2	0.3	0.5	0.7	0.9	1.0
6	20	IVIM 1	0.098859	0.197589	0.296066	0.491778	0.685109	0.875322	0.969080
		IVIM 2	0.098848	0.197568	0.296031	0.491707	0.684985	0.875127	0.968844
	50	IVIM 1	0.098818	0.197502	0.295924	0.491479	0.684561	0.874407	0.967929
		IVIM 2	0.098816	0.197499	0.295918	0.491467	0.684541	0.874375	0.967890
	100	IVIM 1	0.098671	0.197193	0.295419	0.490433	0.682688	0.871347	0.964127
		IVIM 2	0.098670	0.197192	0.295418	0.490430	0.682682	0.871338	0.964117
	200	IVIM 1	0.098646	0.197140	0.295333	0.490254	0.682368	0.870825	0.963479
		IVIM 2	0.098645	0.197139	0.295333	0.490253	0.682366	0.870822	0.963476
	500	IVIM 1	0.098630	0.197107	0.295281	0.490145	0.682174	0.870509	0.963087
		IVIM 2	0.098630	0.197107	0.295281	0.490145	0.682174	0.870508	0.963086
	1000	IVIM 1	0.098625	0.197097	0.295263	0.490109	0.682109	0.870403	0.962956
		IVIM 2	0.098625	0.197097	0.295263	0.490109	0.682109	0.870403	0.962955
		Numerical	0.098620	0.197086	0.295246	0.490073	0.682044	0.870297	0.962824

Table 3
Results of numerical, IVIM Type I and IVIM Type II solutions for different values of initial headways in Mode 2.

IVIM conditions		Type of solution	Initial headway						
Order	Subintervals		0.1	0.2	0.3	0.5	0.7	0.9	1.0
6	20	IVIM 1	0.088080	0.175865	0.263067	0.434665	0.601014	0.760724	0.837769
		IVIM 2	0.088042	0.175792	0.262967	0.434555	0.601024	0.761147	0.838612
	50	IVIM 1	0.087498	0.174689	0.261276	0.431536	0.596364	0.754337	0.830433
		IVIM 2	0.087537	0.174772	0.261409	0.431822	0.596935	0.755491	0.832093
	100	IVIM 1	0.087303	0.174294	0.260673	0.430485	0.594800	0.752183	0.827951
		IVIM 2	0.087355	0.174403	0.260848	0.430842	0.595479	0.753494	0.829801
	200	IVIM 1	0.087204	0.174096	0.260371	0.429957	0.594015	0.751100	0.826702
		IVIM 2	0.087261	0.174214	0.260559	0.430339	0.594733	0.752475	0.828633
	500	IVIM 1	0.087145	0.173977	0.260189	0.429640	0.593542	0.750449	0.825949
		IVIM 2	0.087204	0.174098	0.260383	0.430033	0.594281	0.751856	0.827925
	1000	IVIM 1	0.087126	0.173937	0.260129	0.429534	0.593385	0.750231	0.825698
		IVIM 2	0.087185	0.174059	0.260324	0.429930	0.594129	0.751649	0.827688
		Numerical	0.087166	0.174021	0.260266	0.429830	0.593987	0.751472	0.827500

obtained for different values of initial headways changing in the interval [0, 1] with a step size of 0.1. The order of solution is at most six and numbers of subintervals are selected as 20, 50, 100, 200, 500, 1000 for each pre-defined initial headway. The results are shown in Tables 2–5.

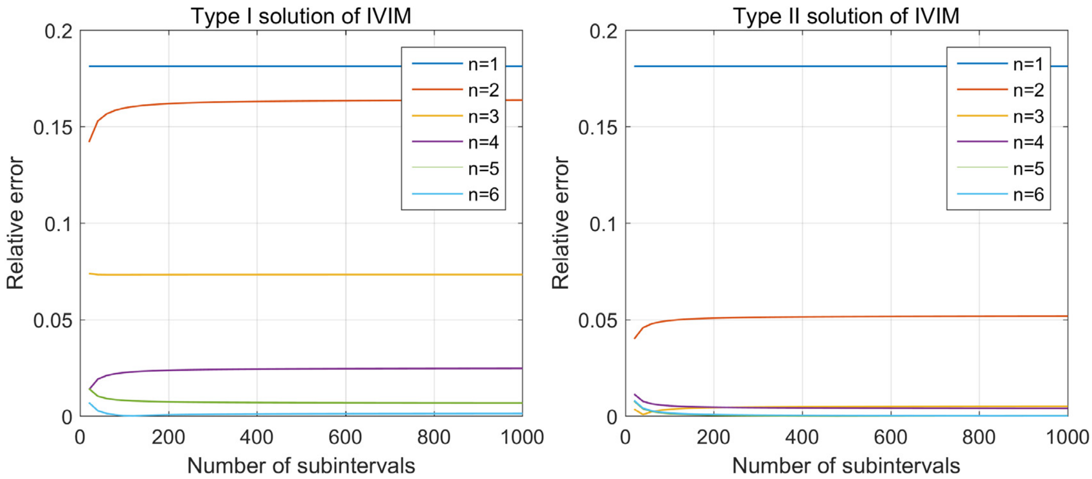

Fig. 3. The relative errors between IVIM and numerical solutions with Type I and Type II solutions of the IVIM for Mode 3.

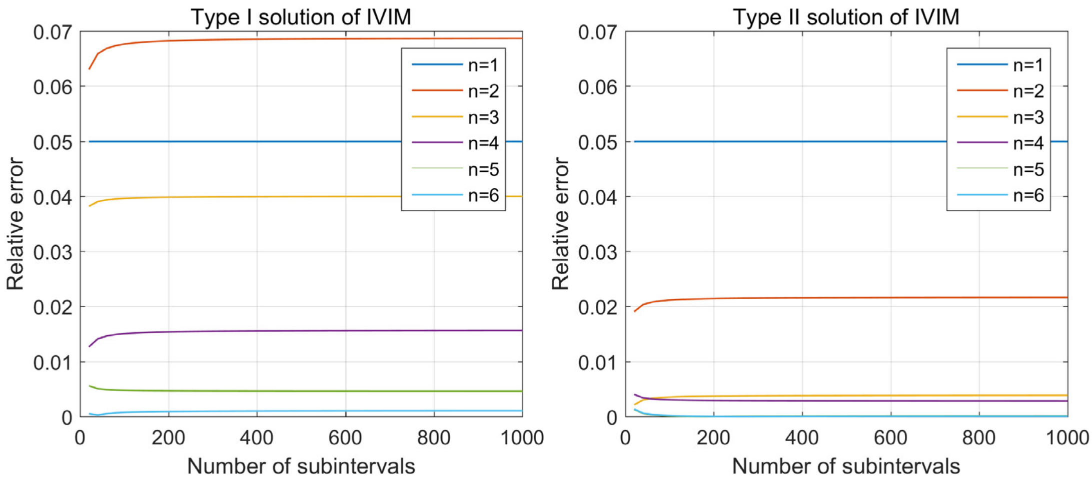

Fig. 4. The relative errors between IVIM and numerical solutions with Type I and Type II solutions of the IVIM for Mode 4.

Maximum number of subintervals is selected as 1000 for each order of solution. However, it can be seen from Figs. 1–4 that after approximately 200 subintervals the relative error becomes undistinguishable due to very small changes in the result produced by the method. This behavior can also be observed from Tables 2–5 in which a gradual convergence is still in progress. In Fig. 5, headway deviation is compared with time in $\eta - t$ diagrams for different modes with type 1 solution of IVIM and also numerical solution provided by Mathematica. The figures depict that calculated headway deviations get better convergence with numerical solution while the order of solution is increasing. Besides, it may be concluded that further improvement is still required for IVIM type 1 solution which is two order higher than type 2 solution shown in Fig. 6.

In Fig. 6, numerical results are compared with IVIM type 2 solution. From the graphs one can conclude that a fourth order solution is enough for acceptable accuracy in the solution of the problem. Better convergence is observed for type 2 formulation of IVIM. This is an expected behavior due to the better representation of the linear operator in type 2 formulation. Computed headway deviations have better agreement with the numerical solution while the order of solution is increasing.

Table 4
Results of numerical, IVIM Type I and IVIM Type II solutions for different values of initial headways in Mode 3.

IVIM conditions		Type of solution	Initial headway						
Order	Subintervals		0.1	0.2	0.3	0.5	0.7	0.9	1.0
6	20	IVIM 1	0.149669	0.296913	0.439515	0.703898	0.933957	1.128179	1.212830
		IVIM 2	0.149372	0.296338	0.438693	0.702701	0.932562	1.126807	1.211541
	50	IVIM 1	0.152861	0.303075	0.448231	0.715980	0.946872	1.139788	1.223208
		IVIM 2	0.152343	0.302080	0.446830	0.714037	0.944756	1.137850	1.221449
	100	IVIM 1	0.152186	0.301775	0.446399	0.713471	0.944227	1.137439	1.221119
		IVIM 2	0.151676	0.300795	0.445022	0.711566	0.942161	1.135556	1.219414
	200	IVIM 1	0.152514	0.302407	0.447292	0.714701	0.945533	1.138606	1.222159
		IVIM 2	0.151988	0.301398	0.445873	0.712742	0.943412	1.136677	1.220414
	500	IVIM 1	0.152712	0.302789	0.447831	0.715443	0.946319	1.139307	1.222783
		IVIM 2	0.152178	0.301764	0.446390	0.713454	0.944168	1.137352	1.221016
	1000	IVIM 1	0.152778	0.302917	0.448011	0.715691	0.946581	1.139541	1.222991
		IVIM 2	0.152242	0.301887	0.446563	0.713693	0.944421	1.137578	1.221217
		Numerical	0.152310	0.302013	0.446731	0.713915	0.944676	1.137780	1.221360

Table 5
Results of numerical, IVIM Type I and IVIM Type II solutions for different values of initial headways in Mode 4.

IVIM conditions		Type of solution	Initial headway						
Order	Subintervals		0.1	0.2	0.3	0.5	0.7	0.9	1.0
6	20	IVIM 1	0.139836	0.276427	0.406896	0.641338	0.834569	0.988111	1.052013
		IVIM 2	0.139074	0.274979	0.404898	0.638779	0.832234	0.986653	1.051140
	50	IVIM 1	0.140915	0.278468	0.409687	0.644790	0.837581	0.989904	1.053025
		IVIM 2	0.140018	0.276768	0.407354	0.641848	0.834947	0.988289	1.052064
	100	IVIM 1	0.141288	0.279172	0.410647	0.645970	0.838604	0.990511	1.053367
		IVIM 2	0.140353	0.277401	0.408219	0.642915	0.835876	0.988842	1.052376
	200	IVIM 1	0.141477	0.279529	0.411134	0.646566	0.839120	0.990817	1.053540
		IVIM 2	0.140524	0.277725	0.408660	0.643457	0.836346	0.989120	1.052532
	500	IVIM 1	0.141591	0.279744	0.411427	0.646925	0.839430	0.991001	1.053644
		IVIM 2	0.140628	0.277921	0.408928	0.643785	0.836629	0.989287	1.052626
	1000	IVIM 1	0.141629	0.279816	0.411525	0.647045	0.839534	0.991062	1.053679
		IVIM 2	0.140663	0.277987	0.409017	0.643894	0.836724	0.989343	1.052657
		Numerical	0.140700	0.278053	0.409103	0.644004	0.836798	0.989290	1.052570

6. Conclusion

In this study, Jamming Transition Problem (JTP) is solved using Interpolated Variational Iteration Method (IVIM). JTP is modeled via Lorenz system that is a nonlinear non-conservative oscillator. In this model dissipative dynamics of traffic flow can be represented within the framework of the Lorenz scheme. After some rearrangements in Lorenz system a second order and highly nonlinear differential equation is obtained as the governing equation of JTP. The problem was previously solved using analytical approximation methods which were cited in the text. One of these methods is variational iteration method (VIM) based on variational theory. IVIM provides numerical evaluations of the analytical integrations in the VIM formulation of the problem which reduce the solution time significantly. In the study two different formulations are considered based on the two different assumptions of the linear operator in governing system of equations. Results have shown that better convergence is obtained with formulation type 2 which provides a better description of the linear operator. In this formulation all linear terms are included in the linear operator. As expected better convergence is achieved in this analysis that is shown in

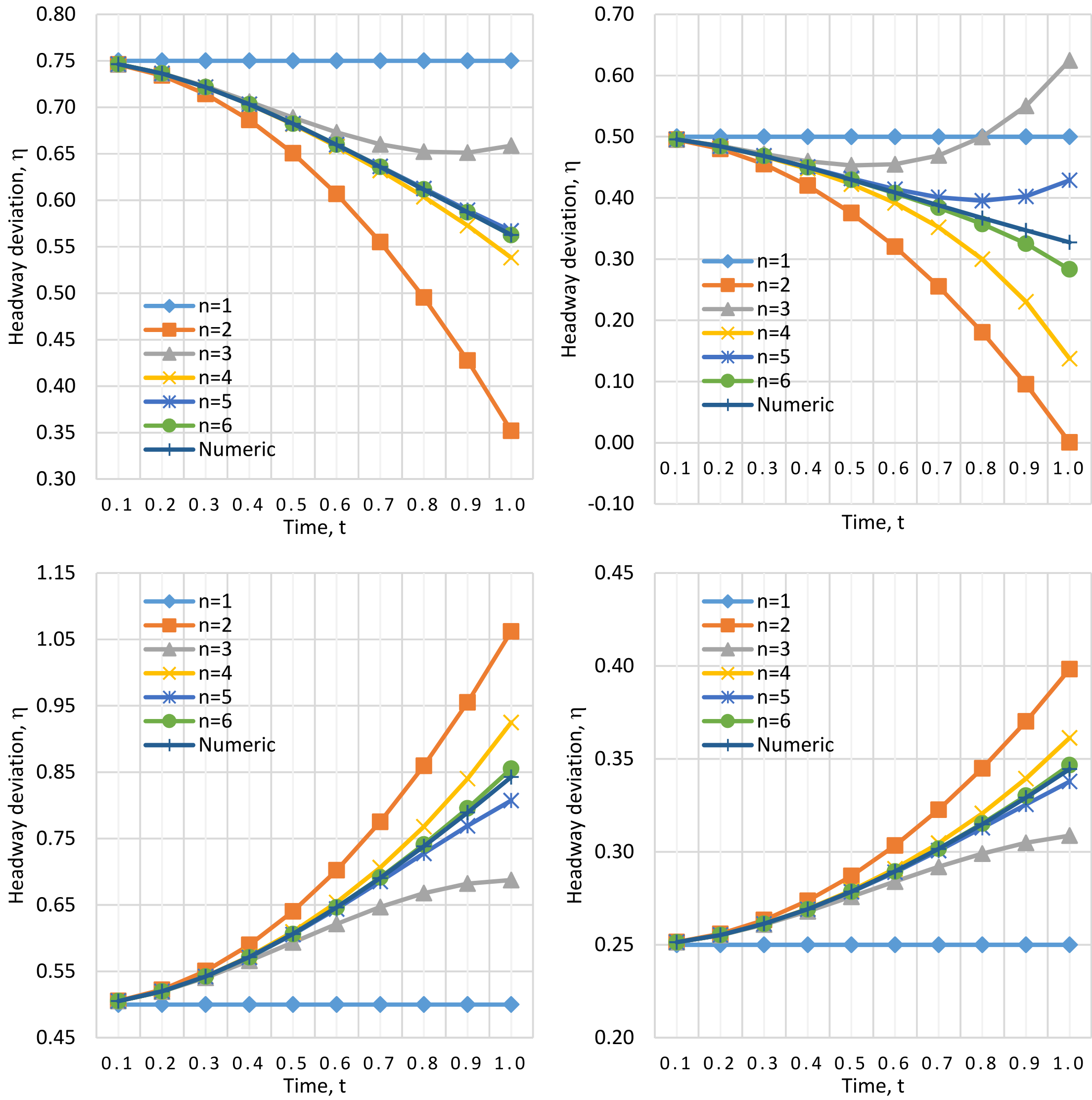

Fig. 5. Variation of headway deviation with time for different modes with IVIM type 1 and numerical solutions.

Fig. 6. A simple definition for linear operator was assumed in formulation type 1. Hence, the solution converges relatively slow when compared to the convergence provided with formulation type 2. For an accurate result some additional iterations of the solution procedure are required with formulation type 1 which lead to a higher order solution. An adequate evaluation of integrals in governing equation required at least 200 subintervals in the analysis of JTP. Numerical integration in IVIM reduces time duration for integration drastically while the solution is still based on an analytical derivation. As a result, IVIM is highly recommended for the solution of problems governed by highly nonlinear differential equations for which there is no available analytical solution.

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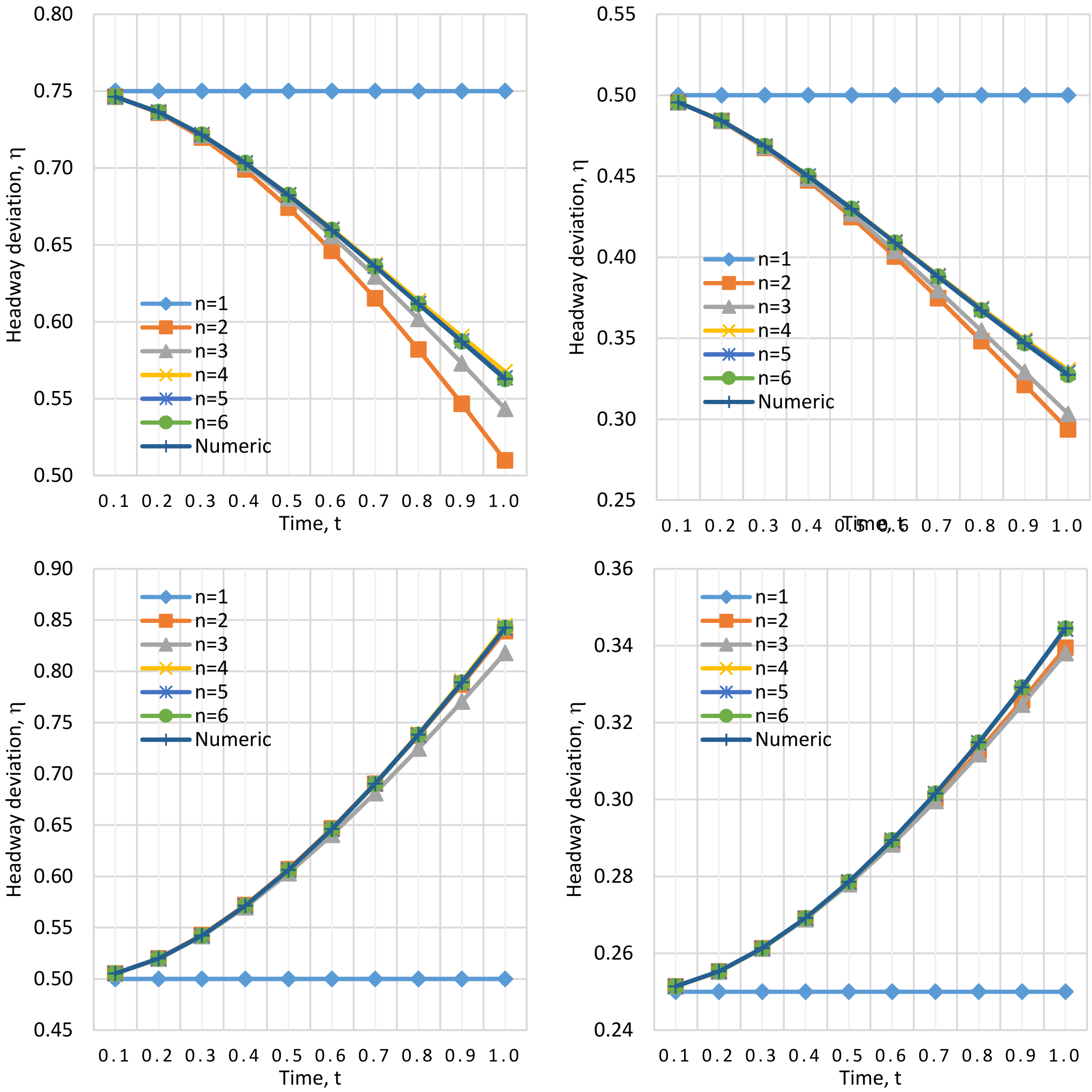

Fig. 6. Variation of headway deviation with time for different modes with IVIM type 2 and numerical solutions.

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